J. London Math. Soc. (2) 67 (2003) 380–400
DOI: 10.1112/S0024610702003897

ESCAPING POINTS OF EXPONENTIAL MAPS

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Abstract

The points which converge to ∞ under iteration of the maps $z \longmapsto \lambda \exp(z)$ for $\lambda \in \mathbb{C}\backslash\{0\}$ are investigated. A complete classification of such 'escaping points' is given: they are organized in the form of differentiable curves called rays which are diffeomorphic to open intervals, together with the endpoints of certain (but not all) of these rays. Every escaping point is either on a ray or the endpoint (landing point) of a ray. This answers a special case of a question of Eremenko. The combinatorics of occurring rays, and which of them land at escaping points, are described exactly. It turns out that this answer does not depend on the parameter λ.

It is also shown that the union of all the rays has Hausdorff dimension 1, while the endpoints alone have Hausdorff dimension 2. This generalizes results of Karpińska for specific choices of λ.

1. *Introduction*

This paper is a contribution to the program to carry results and techniques from the successful theory of iterated polynomials over to the theory of iterated entire maps. One major tool for polynomials are *dynamic rays* introduced by Douady and Hubbard **[4]**; they organize the points which iterate towards ∞, and they allow us to describe the topology and dynamics of Julia set very precisely.

We concentrate on the simplest class of iterated entire maps: these are the maps $z \longmapsto \lambda \exp(z) = \exp(z + \kappa)$ with $\lambda \in \mathbb{C}\backslash\{0\}$ or $\kappa \in \mathbb{C}$. It is our belief that a solid understanding of particular families of maps is a helpful step towards a theory for more general entire maps, and it may suggest results or counterexamples that one can expect in general. Recent progress on the cosine family $z \longmapsto ae^z + be^{-z}$ **[21]** and more general entire maps (see Rottenfußer, work in progress) indicates that the phenomena observed for exponential maps are true in much greater generality. It is well known that the decisive role of critical values for iterated rational maps is, for entire maps, assumed by *singular values*: these are critical values or asymptotic values. One reason why exponential maps are particularly easy to study is that they only have one singular value: the asymptotic value 0.

We furnish a complete classification of points which iterate towards ∞ under $z \longmapsto \lambda \exp(z)$ for a given λ (Corollary 6.9). It turns out that, as in the polynomial case, such points are organized in the form of dynamic rays. There are two new phenomena: dynamic rays on which all points have a certain minimal 'rate of escape', and rays which 'land' at points so that the landing points themselves escape. We describe precisely the set of rays for which this occurs; it is related to unbounded combinatorics. Any escaping point is either part of a dynamic ray, or it is the landing point of a unique ray. This gives a very precise answer to a question of Eremenko **[5]** in the particular case of our maps: the original question was whether any escaping point (of any iterated entire map) can be connected to ∞ by a curve

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Received 24 May 2000; revised 11 June 2001.

2000 *Mathematics Subject Classification* 30D05, 33B10, 37B10, 37B45, 37C35, 37C45, 37C70, 37F10, 37F20, 37F35.

through escaping points. The answer in our case is yes, and the curve is unique. We should note that our classification does not involve the parameter λ or κ (there are natural exceptions though if the singular value 0 escapes, and these are analogous to the exceptions known from the polynomial theory).

Our classification leads to an immediate generalization of Karpińska's paradox [**9**], which shows that the set of rays has Hausdorff dimension 1, while the set of escaping landing points has dimension 2 (and yet all the escaping landing points can be connected by disjoint rays to ∞, using only some of the rays).

There is a collection of previous papers on the set of escaping points of exponential maps: in [**2**], rays are constructed (under the name of 'hairs') for *bounded regular* combinatorics (where 'regular' means that the combinatorics should not contain any entry 0, see the discussion in Section 6). Devaney and Krych [**3**] discuss rays with unbounded combinatorics which satisfy certain bounds depending on λ, but only for λ real. Viana da Silva [**19**] does the same for arbitrary λ and proves that rays are infinitely differentiable. Our preprint [**18**, **22**] classifies all escaping points with bounded combinatorics (and discusses landing properties of periodic and preperiodic rays).

There is a natural overlap with all these papers. In particular, the initial constructions (using what we call 'static partitions') are the same. The main differences from these previous papers are the following: we turn the construction of certain escaping points into a complete classification, and we show that the growth conditions for the combinatorics is independent of λ. A major difference in particular from [**2**] is that we do not force our rays to remain within the strips used in the initial construction (which we view as an unnatural condition).

In [**1**, **2**, **7**], the investigation of the parameter space of iterated exponential maps was begun, by analogy to the well understood Mandelbrot set. More recently, this program was taken further in [**16**]: in particular, it was shown that the set of parameters for which the singular value escapes with bounded combinatorics is organized in the form of rays in parameter space. These methods allow us to transfer our classification results to parameter space [**20**]. A complete classification of exponential maps with attracting dynamics can be found in [**17**]. We should also mention several papers which discuss the dynamics of exponential maps from a measurable point of view, in particular Lyubich [**10**], Rees [**14**], Eremenko and Lyubich [**8**], and McMullen [**11**]. See Section 7 for a brief discussion.

1.1. *Some conventions and notation*

We will usually write the maps $\lambda \exp(z)$ as $\exp(z+\kappa)$, where κ fixes a particular choice of $\log\lambda$ (and $\lambda = \exp(\kappa)$ can always be reconstructed). While the exact choice of the logarithm is in principle inessential, we have written our estimates and combinatorics for $|\mathrm{Im}(\kappa)| \leqslant \pi$; this is no loss of generality. Although many of our constructions depend on κ, we will usually suppress that is the notation.

We use the notation $E_\lambda(z) := \exp(z+\kappa) = \lambda\exp(z)$, $\mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}$, $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ and $\mathbb{C}' := \mathbb{C}^*\backslash\mathbb{R}^-$. The principal branch of the logarithm in $\mathbb{C}'$ will be denoted Log. We will often need $F(t) := \exp(t) - 1$ as a comparison function (usually for real values $t > 0$).

2. *Escaping points and symbolic dynamics*

LEMMA 2.1 (real parts of escaping orbits). *If (z_k) is an orbit for which $|z_k| \to \infty$ as $k \to \infty$, then $\mathrm{Re}(z_k) \to +\infty$.*

Proof. This follows from $|z_{k+1}| = |\lambda| \exp(\mathrm{Re}(z_k))$. □

DEFINITION 2.2 (escaping point). A point $z \in \mathbb{C}$ with $\mathrm{Re}(E_\lambda^{\circ n}(z)) \to +\infty$ as $n \to +\infty$ will be called an *escaping point*; its orbit will be called an *escaping orbit*.

For $j \in \mathbb{Z}$, we define the strips

$$R_j := \{z \in \mathbb{C} : -\mathrm{Im}(\kappa) - \pi + 2\pi j < \mathrm{Im}(z) < -\mathrm{Im}(\kappa) + \pi + 2\pi j\};$$

then $E_\lambda : R_j \to \mathbb{C}'$ is a conformal isomorphism for every j. The assumption $|\mathrm{Im}(\kappa)| \leqslant \pi$ implies that the singular value 0 is always in $\overline{R}_0$. The union of all R_j is a partition of the complex plane; the boundaries are the preimages of the negative real axis, since E_λ maps each strip R_j to the slit complex plane $\mathbb{C}'$. The inverse of E_λ mapping $\mathbb{C}'$ to R_j will be denoted by $L_{\kappa,j}$, so that $L_{\kappa,j}(z) = \mathrm{Log}\, z - \kappa + 2\pi i j$. As a consequence, $R_j \subseteq E_\lambda(R_k)$ for every $j \neq 0$ and every k. Note that R_0 is the only strip having nonempty intersection with the image of the boundary of an arbitrary strip.

The strips are analogous to 'sectors' for polynomials: they are limits of polynomial sectors of angles $2\pi/d$ with vertices at $-d$, which come up in the study of 'unicritical polynomials' $z \longmapsto \lambda(1 + z/d)^d$ with critical points at $-d$ and critical values at 0.

DEFINITION 2.3 (external address). Let $\mathcal{S} := \{(s_1 s_2 s_3, \ldots) : \text{all } s_k \in \mathbb{Z}\}$ be the space of sequences over the integers, and let σ be the shift map on $\mathcal{S}$. We will often use the abbreviation $\underline{s} = (s_1 s_2 s_3 \ldots)$. For any $z \in \mathbb{C}$ with $E_\lambda^{\circ n}(z) \notin \mathbb{R}^-$ for all $n \in \mathbb{N}$, the *external address* $S(z) \in \mathcal{S}$ is the sequence of numbers of the strips containing $z, E_\lambda(z), E_\lambda^{\circ 2}(z), \ldots$.

External addresses are defined in particular for all orbits which always remain in the right half plane.

External addresses correspond to 'binary expansions' of external angles of monic quadratic polynomials or base d expansions in degree d, written as $0.s_1 s_2 s_3 \ldots$ with $-d/2 < s_i \leqslant d/2$ (instead of the usual $0 \leqslant s_i \leqslant d - 1$, to make the discontinuity disappear in the limit $d \to \infty$).

The idea of defining a partition by considering the preimages of the negative real axis can be found in [**2**]; we call such a partition a *static partition*, as opposed to various dynamically more natural partitions introduced in [**18**, **22**]. Sometimes the word 'itinerary' is used for our external addresses; we reserve that word for dynamically natural partitions, by analogy with polynomials.

We will call a sequence $\underline{s} \in \mathcal{S}$ *exponentially bounded* if there are $A, x \geqslant 0$ such that $|s_k| \leqslant AF^{\circ(k-1)}(x)$ for all $k \geqslant 1$. This condition is preserved under the shift map, but the constants change: if $\underline{s}' = \sigma(\underline{s})$, then $|s'_k| \leqslant AF^{\circ(k-1)}(F(x))$. (Unbounded itineraries were considered in [**3**] for real λ and in [**19**] in general. In these papers, the exponential bounds depend on $|\lambda|$, but it turns out that this is inessential. Recently, this has also been observed by Devaney (personal communication).)

LEMMA 2.4 (external addresses are exponentially bounded). *For any exponential map E_λ, any orbit (z_k) satisfies, for all $k \geqslant 1$, the bound*

$$\max\{\mathrm{Re}(z_k), |\mathrm{Im}(z_k)|\} \leqslant |z_k| < F^{\circ(k-1)}(|z_1| + \delta),$$

where $\delta \geqslant 2\pi$ is such that $|\lambda| < e^\delta - (\delta + 1)$. In particular, any orbit which avoids $\mathbb{R}^-$ has an exponentially bounded external address.

Proof. For all z_k, we can estimate

$$\begin{aligned}|z_{k+1}| + \delta &= |\lambda| \exp(\mathrm{Re}(z_k)) + \delta \leqslant |\lambda| \exp|z_k| + \delta \\ &< \exp(|z_k| + \delta) - (\delta + 1)\exp|z_k| + \delta \leqslant \exp(|z_k| + \delta) - 1 \\ &= F(|z_k| + \delta).\end{aligned}$$

Induction yields $|z_k| + \delta < F^{\circ(k-1)}(|z_1| + \delta)$ and proves the first claim. If the orbit avoids $\mathbb{R}^-$, then the itinerary $s_1 s_2 s_3 \ldots$ is defined and we have $2\pi|s_k| \leqslant |\mathrm{Im}(z_k)| + 2\pi \leqslant |z_k| + \delta < F^{\circ(k-1)}(|z_1| + \delta)$. □

3. *Tails of dynamic rays*

In this section, we show that the set of escaping points with given exponentially bounded external address $\underline{s} \in \mathcal{S}$ contains 'tails of dynamic rays', which are curves with sufficiently large positive real parts. In [**2**, **19**], similar objects have been examined using the term (tails of) hairs. However, in [**2**], only the special case of bounded external addresses without entry 0 was treated; in [**19**], the goal was to prove that the rays are infinitely differentiable.

DEFINITION 3.1 (tail of ray). A *ray tail* for E_λ with external address $\underline{s} \in \mathcal{S}$ is an injective curve

$$g_{\underline{s}} : [\tau, \infty) \to \mathbb{C}$$

(for some $\tau \in \mathbb{R}$) satisfying the following conditions: each point on the curve is an escaping point, and has the external address $\underline{s}$ and $\lim_{t\to+\infty} \mathrm{Re}(g_{\underline{s}}(t)) = +\infty$.

Tails of rays lie entirely in the Julia set of E_λ: this follows from the classification of Fatou components by Eremenko and Lyubich [**6**, **7**, **8**]; see also [**1**, Theorems 6 and 7]. An elementary argument uses the escaping condition $\lim_{n\to+\infty} \mathrm{Re}(E_\lambda^{\circ n}(g_{\underline{s}}(t))) = +\infty$: there is strong expansion for all orbits which start near the tail of a ray, so there will always be nearby orbits which map far into a left half plane after many iterations and then get close to the origin; this is incompatible with locally uniform convergence in the Fatou set.

Every ray tail satisfies a bound in the vertical direction depending only on the first entry in $\underline{s}$. This will not necessarily be so for the rays introduced in Section 4.

PROPOSITION 3.2 (existence of tails of rays). *For every $\kappa \in \mathbb{C}$ and every exponentially bounded sequence $\underline{s} \in \mathcal{S}$, there is a ray tail with external address $\underline{s}$.*

The rest of this section is devoted to the proof of this proposition. The proof consists in constructing a map $g_{\underline{s}}(t)$ conjugating the dynamics of E_λ on a curve to the dynamics of $F : t \longmapsto e^t - 1$ on some right end of $\mathbb{R}$. Recall that in the polynomial case, the conjugation to $z \longmapsto z^d$ (the Riemann map) is used to define dynamic rays. We will use a similar approach here, but we cannot define it on any open set. Instead, using ideas from [**2**, **19**], we define inductively maps $g_{\underline{s}}^n$ on right ends of $\mathbb{R}$ as follows for $n \in \mathbb{N}$:

$$g_{\underline{s}}^n(t) := L_{\kappa,s_1} \circ \ldots \circ L_{\kappa,s_n} \circ F^{\circ n}(t). \tag{1}$$

We will show below that there is a $t^* \in \mathbb{R}$ such that these maps are defined for all $t \geqslant t^*$ independently of n. The first lemma does not require any bound on $\underline{s}$.

LEMMA 3.3 (bound on real parts). *For every $K > 0$ and for every $\lambda \in \mathbb{C}^*$ with $|\kappa| = |\log\lambda| \leqslant K$, every $n \in \mathbb{N}$, and every external address $\underline{s}$, the function $g_{\underline{s}}^n$ is defined for all $t \geqslant 2\log(K+3)$ and satisfies* $\mathrm{Re}(g_{\underline{s}}^n(t)) \geqslant t-(K+2)$ *and* $g_{\underline{s}}^n(t) = \mathrm{Log}(g_{\sigma(\underline{s})}^{n-1}(F(t))) - \kappa + 2\pi i s_1$.

Proof. We start an induction with $g_{\underline{s}}^0(t) = t$. It is defined for all real t and satisfies $\mathrm{Re}(g_{\underline{s}}^0(t)) > t-(K+2)$. For $n \geqslant 1$, the recursive relation $g_{\underline{s}}^n(t) = L_{\kappa,s_1}(g_{\sigma(\underline{s})}^{n-1}(F(t)))$ is built into the definition. Therefore,

$$\begin{aligned}
\mathrm{Re}(g_{\underline{s}}^n(t)) &= \mathrm{Re}(\mathrm{Log}(g_{\sigma(\underline{s})}^{n-1}(F(t))) - \kappa + 2\pi i s_1) \\
&= \mathrm{Re}(\mathrm{Log}(g_{\sigma(\underline{s})}^{n-1}(F(t)))) - \mathrm{Re}(\kappa) = \log|g_{\sigma(\underline{s})}^{n-1}(F(t))| - \mathrm{Re}(\kappa) \\
&\geqslant \log \mathrm{Re}(g_{\sigma(\underline{s})}^{n-1}(F(t))) - \mathrm{Re}(\kappa) \geqslant \log(F(t)-(K+2)) - |\kappa| \\
&= \log(e^t-(K+3)) - |\kappa| = t - \log\left(\frac{1}{1-(K+3)/e^t}\right) - |\kappa| \\
&> t - \frac{(K+3)/e^t}{1-(K+3)/e^t} - K = t - \frac{1}{e^t/(K+3)-1} - K \\
&\geqslant t - \frac{1}{K+2} - K > t-(K+2).
\end{aligned}$$

Therefore, $g_{\underline{s}}^{n+1}(t) = L_{\kappa,s_1}(g_{\sigma(\underline{s})}^n(F(t)))$ is defined for all $t \geqslant 2\log(K+3)$ as well, which concludes the inductive proof. □

PROPOSITION 3.4 (parametrization of ray tails). *Fix an exponentially bounded sequence $\underline{s} := (s_1, s_2, \ldots) \in \mathcal{S}$ and a constant $K > 0$ and let $\lambda \in \mathbb{C}^*$ be a parameter with $|\kappa| = |\log\lambda| \leqslant K$. Then there is a $t^* \in \mathbb{R}$ such that the sequence of functions $g_{\underline{s}}^n(t)$ is defined for $t \geqslant t^*$ and converges uniformly in t to a limit function $g_{\underline{s}}(t)$. This function is injective and continuous in t and depends for fixed $t \geqslant t^*$ analytically on κ. It satisfies the functional equation $E_\lambda(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$. Moreover,*

$$g_{\underline{s}}(t) = t - \kappa + 2\pi i s_1 + r_{\underline{s}}(t) \qquad |r_{\underline{s}}(t)| = O(e^{-t}). \tag{2}$$

If $|s_k| < AF^{\circ(k-1)}(x)$ for all $k \geqslant 2$ with $A \geqslant 1/2\pi$ and $x \geqslant 0$, then $t^* := x + 2\log(K+3)$ is a valid choice, and

$$|r_{\underline{s}}(t)| < 2e^{-t}(K+2+2\pi|s_2|+2\pi AC)$$

for a universal constant $C > 0$.

REMARK 3.5. For all bounded $\underline{s}$, one can take $t^* = 2\log(K+3)$, as is readily verified in the proof.

Proof of Proposition 3.4. Let $t^* := x + 2\log(K+3)$ as in the claim. For some fixed $t \geqslant t^*$, let $t_k := F^{\circ k}(t)$. It is then easy to verify by induction that

$$t_k - (K+2) > e^{k-1} \quad \text{and} \quad t_k - (K+2) > K + 1 + F^{\circ k}(x) \tag{3}$$

for $k \geqslant 1$. We will twice need the estimate

$$|\log(F(t)) - t| = |\log(1-e^{-t})| < 2e^{-t} < 1, \tag{4}$$

since, for $0 < \varepsilon < 1$, $|\log(1-\varepsilon)| < \varepsilon/(1-\varepsilon)$ and $t > 2\log 3$. We have

$$g_{\underline{s}}^{n+1}(t) - g_{\underline{s}}^n(t) = L_{\kappa,s_1} \circ \ldots \circ L_{\kappa,s_n} \circ L_{\kappa,s_{n+1}}(F(t_n)) - L_{\kappa,s_1} \circ \ldots \circ L_{\kappa,s_n}(t_n).$$

Lemma 3.3 gives, for $1 \leqslant k \leqslant n-1$,

$$\mathrm{Re}(L_{\kappa,s_{k+1}} \circ \ldots \circ L_{\kappa,s_n} \circ L_{\kappa,s_{n+1}}(F(t_n))) = \mathrm{Re}(g^{n+1-k}_{\sigma^k(\underline{s})}(t_k)) \geqslant t_k - (K+2)$$

and

$$\mathrm{Re}(L_{\kappa,s_{k+1}} \circ \ldots \circ L_{\kappa,s_n}(t_n)) = \mathrm{Re}(g^{n-k}_{\sigma^k(\underline{s})}(t_k)) \geqslant t_k - (K+2).$$

Therefore, the two points $L_{\kappa,s_{k+1}} \circ \ldots \circ L_{\kappa,s_n} \circ L_{\kappa,s_{n+1}}(F(t_n))$ and $L_{\kappa,s_{k+1}} \circ \ldots \circ L_{\kappa,s_n}(t_n)$ can be connected by a straight line segment in the half plane $\{z \in \mathbb{C} \colon \mathrm{Re}(z) > t_k - (K+2)\}$.

Since, by (4),

$$|L_{\kappa,s_{n+1}}(F(t_n)) - t_n| = |\log F(t_n) - \kappa + 2\pi i s_{n+1} - t_n| \leqslant K + 1 + 2\pi A F^{\circ n}(x)$$

and the logarithms in the definitions of $g^{n+1}_{\underline{s}}(t)$ and $g^{n}_{\underline{s}}(t)$ use the same branches because they are applied only to arguments within the right half plane, we can estimate, for $n \geqslant 2$,

$$\begin{aligned} |g^{n+1}_{\underline{s}}(t) - g^{n}_{\underline{s}}(t)| &\leqslant (K + 1 + 2\pi A F^{\circ n}(x)) \left(\prod_{k=1}^{n} (t_k - (K+2)) \right)^{-1} \\ &< \frac{K + 1 + 2\pi A F^{\circ n}(x)}{t_n - (K+2)} (t_1 - (K+2))^{-1} \prod_{k=2}^{n-1} e^{-(k-1)} \\ &< 2\pi A \, \frac{e^{-t}}{1 - (K+3)e^{-t}} \prod_{k=1}^{n-2} e^{-k} \\ &< 2\pi A \cdot 2e^{-t} \prod_{k=1}^{n-2} e^{-k} \qquad (5) \end{aligned}$$

(in these three inequalities, we used $|L'_{\kappa,s_k}(z)| = 1/|z| \leqslant 1/\mathrm{Re}(z)$ for any k, then both inequalities in (3), and finally $(1-(K+3)e^{-t})^{-1} < 2$ for $t \geqslant 2\log(K+3)$ and $K > 0$).

Similarly, we have

$$|g^{2}_{\underline{s}}(t) - g^{1}_{\underline{s}}(t)| < \frac{K + 1 + 2\pi |s_2|}{t_1 - (K+2)} < 2e^{-t}(K + 1 + 2\pi|s_2|). \qquad (6)$$

We see that the functions $g^n_{\underline{s}}(t)$ converge to a limit $g_{\underline{s}}(t)$ as $n \to \infty$. The convergence is uniform in t and κ provided $t > t^*$ and $|\kappa| \leqslant K$. This proves continuity of the limit $g_{\underline{s}}$. Moreover, since all $g^n_{\underline{s}}$ are holomorphic in κ for fixed t, so is the limit $g_{\underline{s}}$ by the Weierstrass theorem. The functional equation $E_\lambda(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$ follows from $E_\lambda(g^n_{\underline{s}}(t)) = L_{\kappa,s_2} \circ \ldots \circ L_{\kappa,s_n}(F^{\circ n}(t)) = L_{\kappa,s_2} \circ \ldots \circ L_{\kappa,s_n}(F^{\circ(n-1)}(F(t))) = g^{n-1}_{\sigma(\underline{s})}(F(t))$ for every $t \geqslant x + 2\log(K+3)$: $g_{\sigma(\underline{s})}(t')$ is defined for $t' \geqslant F(x) + 2\log(K+3)$, and it is easy to check that $t \geqslant x + 2\log(K+3)$ implies that $F(t) \geqslant F(x) + 2\log(K+3)$.

Using (5), (6) and then (4), we arrive at

$$\begin{aligned} |r_{\underline{s}}(t)| &< |g_{\underline{s}}(t) - g^2_{\underline{s}}(t)| + |g^2_{\underline{s}}(t) - g^1_{\underline{s}}(t)| + |g^1_{\underline{s}}(t) - (t - \kappa + 2\pi i s_1)| \\ &< 2e^{-t} \left(2\pi A \sum_{n=2}^{\infty} \prod_{k=1}^{n-2} e^{-k} + K + 1 + 2\pi|s_2| \right) + |\log(F(t)) - t| \\ &< 2e^{-t}(2\pi A C + K + 2 + 2\pi|s_2|) \end{aligned}$$

where $C = \sum_{n=2}^{\infty} \prod_{k=1}^{n-2} e^{-k} \approx 1.42$ is a universal constant.

Injectivity of $g_{\underline{s}}$ follows like this: if $g_{\underline{s}}(t') = g_{\underline{s}}(t'')$ for $t'' > t' \geqslant t^*$, then the functional equation implies that $g_{\sigma^n(\underline{s})}(F^{\circ n}(t')) = g_{\sigma^n(\underline{s})}(F^{\circ n}(t''))$ for all $n \geqslant 0$ with the bounds

$$\begin{aligned} &|g_{\sigma^n(\underline{s})}(F^{\circ n}(t')) - (F^{\circ n}(t') - \kappa + 2\pi i s_{n+1})| \\ &\quad = |r_{\sigma^n(\underline{s})}(F^{\circ n}(t'))| < 2(2\pi AC + K + 2 + 2\pi|s_{n+2}|)\exp(-F^{\circ n}(t')) \\ &\quad < 2\frac{2\pi AC + K + 2 + 2\pi|s_{n+2}|}{F^{\circ(n+1)}(t')}, \end{aligned}$$

and similarly for t''. Since $|s_{n+2}| < AF^{\circ(n+1)}(x) < AF^{\circ(n+1)}(t')$, the right hand side is bounded as $n \to \infty$ while $|F^{\circ n}(t') - F^{\circ n}(t'')| \to \infty$. This is a contradiction. □

Proof of Proposition 3.2. The function $g_{\underline{s}}(t)$ defined in Proposition 3.4 is constructed so that it parametrizes a ray tail with the given external address $\underline{s}$: Lemma 3.3 gives $\text{Re}(E_\lambda^{\circ n}(g_{\underline{s}}(t))) = \text{Re}(g_{\sigma^n(\underline{s})}(F^{\circ n}(t))) \geqslant F^{\circ n}(t) - (K+2)$, which shows that $z = g_{\underline{s}}(t)$ is an escaping point. Moreover, by (2), $\text{Re}(g_{\underline{s}}(t)) \to \infty$ as $t \to +\infty$. The imaginary parts inherit their bounds from the strip R_{s_1}. Hence $g_{\underline{s}}(t)$ is indeed a ray tail with external address $\underline{s}$. □

We will call the variable t the *potential* (or *escape rate*) of the point $g_{\underline{s}}(t)$. Although there is no direct relation to potential theory on open domains, these names are intended to indicate the relation to dynamic rays of polynomials (which are usually parametrized by potentials).

4. *Dynamic rays*

Our goal in this section is to extend the ray tails to as low potentials t as possible, using the functional equation. They satisfy the relation $E_\lambda(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$ with $F(t) = e^t - 1$. We can try to extend the ray tail by pulling it back by the dynamics: $g_{\underline{s}}(t) = E_\lambda^{-1}(g_{\sigma(\underline{s})}(F(t)))$, where the branch of E_λ^{-1} is chosen so that $E_\lambda^{-1}(g_{\sigma(\underline{s})}(F(t)))$ coincides with $g_{\underline{s}}(t)$ for large t. This helps to define rays at successively smaller arguments.

Here is the reason why we have chosen the function $F(t) = e^t - 1$, rather than e^t which would have yielded slightly easier calculations in the previous section: the point $z = E_\lambda^{-1}(g_{\underline{s}}(t))$ should be on the ray $g_{\underline{s}'}$ for some external address $\underline{s}'$ with $\sigma(\underline{s}') = \underline{s}$, and its potential should be $F^{-1}(t)$ because of the dynamic relation between rays. Had we used e^t instead of $e^t - 1$, then the potential of z should have been $\log(t)$. This can be repeated only finitely often until the logarithm runs out of its domain. (In other words, the orbit of the singular value under exp runs exactly along the curve $(0, \infty) \subset \mathbb{R}$ which we are using as a model for the dynamic ray.) Under $F(t)$, however, any $t > 0$ can be iterated backwards infinitely often and converges to 0. (The choice of $F(t) = e^t - 1$ is largely arbitrary; we could conjugate with any monotonically increasing homeomorphism from $[0, \infty)$ to itself. Our choice is mainly for reasons of convenience, yielding the nice asymptotic form (2) for $g_{\underline{s}}(t)$ for large t.)

It turns out that for every exponentially bounded $\underline{s} \in \mathscr{S}$, there is a minimal potential $t_{\underline{s}} \geqslant 0$ of escaping points with this external address.

DEFINITION 4.1 (minimal potential of external address). For any sequence $\underline{s} = s_1 s_2 s_3 \ldots \in \mathcal{S}$, define its *minimal potential* $t_{\underline{s}} \in [0, \infty]$ to be

$$t_{\underline{s}} = \inf \left\{ t > 0 : \limsup_{k \geqslant 1} \frac{|s_k|}{F^{\circ(k-1)}(t)} = 0 \right\}.$$

Observe that $t_{\sigma(\underline{s})} = F(t_{\underline{s}})$.

THEOREM 4.2 (dynamic rays). (1) *A sequence $\underline{s} \in \mathcal{S}$ is exponentially bounded if and only if $t_{\underline{s}} < \infty$.*

(2) *If the singular value does not escape, then for every exponentially bounded $\underline{s}$ there is an injective curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ consisting of escaping points such that*

$$E_\lambda(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t)) \qquad \textit{for every } t > t_{\underline{s}} \tag{7}$$

(*and $F(t) > t_{\sigma(\underline{s})}$*) *which extends the ray tail for the external address $\underline{s}$ as constructed in Proposition* 3.4. *In particular, the curve $g_{\underline{s}}$ satisfies the asymptotic bounds in equation* (2).

(3) *If the singular value does escape, then the statement is still true for every $\underline{s}$ unless there are $n \geqslant 1$ and $t_0 > F^{\circ n}(t_{\underline{s}})$ such that $0 = g_{\sigma^n(\underline{s})}(t_0)$. For those exceptional $\underline{s}$, there is an injective curve $g_{\underline{s}} : (t^*_{\underline{s}}, \infty) \longrightarrow \mathbb{C}$ with the same properties as before, where $t^*_{\underline{s}}$ is the largest potential which has an $n \geqslant 1$ such that $F^{\circ n}(t^*_{\underline{s}}) = t_0$ and $0 = g_{\sigma^n(\underline{s})}(t_0)$.*

The curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ (or $g_{\underline{s}} : (t^*_{\underline{s}}, \infty) \to \mathbb{C}$) will be called the *dynamic ray at external address* $\underline{s}$. We should note that in the exceptional case, it is no longer true that the $E_\lambda^{\circ k}$-image of the $\underline{s}$-ray maps entirely onto the $\sigma^k(\underline{s})$-ray; it only covers the part above the potential of some postsingular point.

REMARK 4.3. Note that we no longer require that all the points on the ray $g_{\underline{s}}$ have external address $\underline{s}$; this is one major difference from the definition in [**2**]. Since the static partition is dynamically unnatural, there is no reason why dynamic rays should respect it. Only the points at large potentials (on the ray tail) must have external address $\underline{s}$. This is not unnatural in view of Lemma 2.1.

Proof of Theorem 4.2. For $t > 0$, define

$$c(t) := \limsup_{k \geqslant 1} \frac{|s_k|}{F^{\circ(k-1)}(t)}.$$

Clearly, for $t > t'$, we have $F^{\circ(k-1)}(t)/F^{\circ(k-1)}(t') \to \infty$ as $k \to \infty$. Hence, if $c(t) > 0$, then $c(t') = \infty$ for $t' < t$; and if $c(t) < \infty$, then $c(t') = 0$ for $t' > t$ and even $\lim_{k \geqslant 1} |s_k|/F^{\circ(k-1)}(t') = 0$.

If $\underline{s}$ is exponentially bounded, then $|s_k| < AF^{\circ(k-1)}(x)$ for some A and x, so $c(t) = 0$ for $t > x$ and $t_{\underline{s}} < \infty$. Conversely, if $t_{\underline{s}} < \infty$, then for any $\varepsilon > 0$, $|s_k|/F^{\circ(k-1)}(t_{\underline{s}} + \varepsilon) \to 0$ as $k \to \infty$, so $\underline{s}$ is exponentially bounded.

Next show that $g_{\underline{s}}(t)$ exists for $t > t_{\underline{s}}$ using the functional equation $E_\lambda^{\circ n}(g_{\underline{s}}(t)) = g_{\sigma^n(\underline{s})}(F^{\circ n}(t))$ for n sufficiently large: given $t > t_{\underline{s}}$, we need to find an n such that $g_{\sigma^n(\underline{s})}(F^{\circ n}(t))$ is defined. Fix $\varepsilon > 0$. Then $|s_k| < AF^{\circ(k-1)}(t_{\underline{s}} + \varepsilon)$ for some A and all k. From Proposition 3.4, we know that $g_{\sigma^n(\underline{s})}(t')$ exists for $t' \geqslant F^{\circ n}(t_{\underline{s}} + \varepsilon) + 2\log(K + 3)$ with $K = |\kappa|$. If $t > t_{\underline{s}} + 2\varepsilon$, then for n sufficiently large, $F^{\circ n}(t) > F^{\circ n}(t_{\underline{s}} + \varepsilon) + 2\log(K + 3)$; hence $g_{\sigma^n(\underline{s})}(F^{\circ n}(t)) = E_\lambda^{\circ n}(g_{\underline{s}}(t))$ exists and we can define $g_{\underline{s}}(t)$ by iterating E_λ backwards n times. The correct branch of E_λ^{-1} follows by continuity, starting with large values of t. (In view of the functional equation, n can be increased without causing any harm, so the exact choice of n is immaterial.) Therefore, the curve $g_{\underline{s}}$ exists indeed for all $t > t_{\underline{s}}$. Injectivity for $t > t_{\underline{s}}$ is inherited from Proposition 3.4. (It is in this pull-back step that the exception in the statement comes in: if the singular value escapes and is on the dynamic ray that we try to pull back, there is a missing inverse image.) □

Having constructed many escaping points, we can now start to investigate under which conditions an escaping point is on a dynamic ray. The condition in the following theorem might seem rather special, but we will show below that all points on all rays satisfy it in a stronger form.

THEOREM 4.4 (fast escaping points are on ray). *Let* (z_k) *be an escaping orbit within the right half plane and let* $\underline{s}$ *be the external address of* z_1. *If there is a* $t' > t_{\underline{s}}$ *such that* $\mathrm{Re}(z_k) \geqslant F^{\circ(k-1)}(t')$ *for infinitely many* k, *then* $z_1 = g_{\underline{s}'}(t)$ *for some* $t \geqslant t'$ *and an external address* $\underline{s}'$ *which differs from* $\underline{s}$ *only at finitely many entries. If* $t' > K + 2$, *then* $\underline{s}' = \underline{s}$.

Proof. For $k \geqslant 1$, let $t_k > 0$ be such that $F^{\circ(k-1)}(t_k) = \mathrm{Re}(z_k)$. Then, by Lemma 2.4, the sequence t_k is bounded above. Moreover, $|\mathrm{Im}(z_k)| \leqslant 2\pi(|s_k| + 1) < F^{\circ(k-1)}(t')$ for all sufficiently large k. By assumption, infinitely many k satisfy $t_k \geqslant t'$, and for all but finitely many of them we have $|\mathrm{Im}(z_k)| < \mathrm{Re}(z_k)$. In the following, we will look only at such k.

Choose again $K > 0$ such that $|\log \lambda| < K$. There is an $A > 0$ such that $|s_k| < AF^{\circ(k-1)}(t_{\underline{s}} + 1)$, so the ray tails $g^k_{\underline{s}}$ from Proposition 3.4 are defined for all $t > t_{\underline{s}} + 1 + 2\log(K + 3)$; by Lemma 3.3, they satisfy $\mathrm{Re}(g^k_{\underline{s}}(t)) > 0$ if $t > K + 2$, so these ray tails never cross $\mathbb{R}^-$ or the partition boundary $E^{-1}_{\lambda}(\mathbb{R}^-)$.

Suppose first that $\mathrm{Re}(z_k) \geqslant 3$ and $t_k \geqslant \max\{t_{\underline{s}} + 1 + 2\log(K + 3), K + 2\}$ for all $k \geqslant 1$.

By construction, $|z_k - F^{\circ(k-1)}(t_k)| = |\mathrm{Im}(z_k)|$. Pulling the two points z_k and $F^{\circ(k-1)}(t_k)$ back using $L_{\kappa,s_1} \circ \ldots \circ L_{\kappa,s_{k-1}}$, we obtain z_1 and $g^{k-1}_{\underline{s}}(t_k)$, respectively. For $j = 1, 2, \ldots, k$, let $w_j := L_{\kappa,s_j} \circ \ldots \circ L_{\kappa,s_{k-1}}(F^{\circ(k-1)}(t_k))$, so $w_1 = g^{k-1}_{\underline{s}}(t_k)$. Since $\mathrm{Re}(z_k) = F^{\circ(k-1)}(t_k)$, we obtain $|L'_{\kappa,s_{k-1}}(z)| \leqslant 1/\mathrm{Re}(z_k)$ for all z on the vertical line segment between z_k and $F^{\circ(k-1)}(t_k)$, and after $L_{\kappa,s_{k-1}}$ we get the two points z_{k-1} and w_{k-1} with $|z_{k-1} - w_{k-1}| < |\mathrm{Im}(z_k)|/\mathrm{Re}(z_k) < 1$. We maintain the inductive relation $|z_j - w_j| < 2^{-(k-1-j)}$ and $\mathrm{Re}(w_j) \geqslant \mathrm{Re}(z_j) - 2^{-(k-1-j)} \geqslant 2$ for $j = k-1, k-2, \ldots, 1$; hence all the further logarithms contract distances by at least a factor 2, justifying the inductive relation. For $j = 1$, we finally obtain $|z_1 - g^{k-1}_{\underline{s}}(t_k)| < 2^{-(k-2)}$.

Choose $\varepsilon > 0$. Let t be a limit point of the sequence (t_k) (restricted to such k as described above). Clearly $t \geqslant t' > t_{\underline{s}}$. We have $|z_1 - g^{k-1}_{\underline{s}}(t_k)| < \varepsilon$ for large k. For potentials at least t', the approximating curves $g^{k-1}_{\underline{s}}$ converge uniformly to $g_{\underline{s}}$: for t' sufficiently large, this is Proposition 3.4, and for arbitrary compact subintervals of $(t_{\underline{s}}, \infty)$, it follows as in Theorem 4.2. We thus have $|g^{k-1}_{\underline{s}}(t_k) - g_{\underline{s}}(t_k)| < \varepsilon$ (possibly by enlarging k). Finally, for t_k close enough to t, we have $|g_{\underline{s}}(t_k) - g_{\underline{s}}(t)| < \varepsilon$. Combining these estimates, it follows that

$$|z_1 - g_{\underline{s}}(t)| \leqslant |z_1 - g^{k-1}_{\underline{s}}(t_k)| + |g^{k-1}_{\underline{s}}(t_k) - g_{\underline{s}}(t_k)| + |g_{\underline{s}}(t_k) - g_{\underline{s}}(t)| < 3\varepsilon$$

for certain sufficiently large k. Hence $g_{\underline{s}}(t) = z_1$.

If the lower bounds for $\mathrm{Re}(z_k)$ and t_k are not satisfied for all $k \geqslant 1$, there is an

$m \geqslant 1$ such that they are satisfied if z_1 is replaced by z_{m+1}; hence $z_{m+1} = g_{\sigma^m(\underline{s})}(F^{\circ m}(t))$ for some $t \geqslant t'$, and $z_1 = g_{\underline{s}'}(t)$ for some $\underline{s}'$ which can differ from s only in the first m entries.

Finally, if $t' > K+2$, then $\mathrm{Re}(g_{\sigma^k(\underline{s})}[F^{\circ k}(t'), \infty)) > 0$ for $k = 0, 1, 2\ldots$ by Lemma 3.3, so the forward orbit of $g_{\underline{s}}[t', \infty)$ never crosses the partition boundary and $\underline{s}' = \underline{s}$. □

PROPOSITION 4.5 (controlled escape for points on rays). *For every exponentially bounded external address $\underline{s}$ and every $t > t_{\underline{s}}$, the point $g_{\underline{s}}(t)$ satisfies the asymptotic bound $E_\lambda^{\circ k}(g_{\underline{s}}(t)) = F^{\circ k}(t) - \kappa + 2\pi i s_{k+1} + o(1)$ as $k \to \infty$. In particular, it satisfies*

$$\frac{|\mathrm{Im}(E_\lambda^{\circ k}(g_{\underline{s}}(t)))|^p}{\mathrm{Re}(E_\lambda^{\circ k}(g_{\underline{s}}(t)))} \to 0$$

as $k \to \infty$, for every $p > 0$.

Proof. Associated to the exponentially bounded external addresses $\sigma^k(\underline{s})$ are minimal potentials $t_{\underline{s}}^k = F^{\circ k}(t_{\underline{s}})$. By (2) in Proposition 3.4, we have good error bounds for the dynamic rays $g_{\sigma^k(\underline{s})}$ for potentials greater than $t_{\underline{s}}^k + 2\log(K+3)$, where $K \geqslant |\kappa|$. Clearly, for any $t > t_{\underline{s}}$, there is a $k_0 = k_0(t)$ such that $F^{\circ k}(t) > t_{\underline{s}}^k + 2\log(K+3)$ for $k \geqslant k_0$.

Since $|s_k| < AF^{\circ(k-1)}(t_{\underline{s}} + \varepsilon)$ for any $\varepsilon > 0$ and some $A > 0$ depending on ε, we have for $k \geqslant k_0$ by Proposition 3.4

$$E_\lambda^{\circ k}(g_{\underline{s}}(t)) = g_{\sigma^k(\underline{s})}(F^{\circ k}(t)) = F^{\circ k}(t) - \kappa + 2\pi i s_{k+1} + r_{\sigma^k(\underline{s})}(F^{\circ k}(t))$$

with

$$\begin{aligned} |r_{\sigma^k(\underline{s})}(F^{\circ k}(t))| &< 2\exp(-F^{\circ k}(t)) \cdot (K + 2 + 2\pi|s_{k+2}| + 2\pi AC) \\ &< 2\frac{C' + 2\pi AF^{\circ(k+1)}(t_{\underline{s}} + \varepsilon)}{F^{\circ(k+1)}(t)}, \end{aligned} \tag{8}$$

where C is the universal constant from Proposition 3.4 and $C' := K + 2 + 2\pi AC$ depends only on κ and $\underline{s}$, but not on k. If $t_{\underline{s}} + \varepsilon < t$, then the right-hand side of (8) tends to zero as $k \to \infty$ (even extremely rapidly). It follows that, along the orbit of $g_{\underline{s}}(t)$, the real parts grow like $F^{\circ k}(t)$, while the imaginary parts are bounded in absolute value by the asymptotically much smaller quantity $2\pi AF^{\circ k}(t_{\underline{s}} + \varepsilon)$.

More precisely,

$$\log\left(\frac{|\mathrm{Im}(E_\lambda^{\circ k}(g_{\underline{s}}(t)))|^p}{\mathrm{Re}(E_\lambda^{\circ k}(g_{\underline{s}}(t)))}\right) < pF^{\circ(k-1)}(t_{\underline{s}} + \varepsilon) - F^{\circ(k-1)}(t) + O(1) \to -\infty,$$

which settles the last claim. □

5. *Eventually horizontal escape*

For $R \in \mathbb{R}$, we define the right half planes $\mathbb{H}_R := \{z \in \mathbb{C} : \mathrm{Re}(z) > R\}$.

LEMMA 5.1 (exponential separation of orbits). *Let $R > \log(\pi) - \mathrm{Re}(\kappa)$ be positive. Suppose that (z_k) and (w_k) are two escaping orbits for E_λ which are completely contained within the right half plane $\mathbb{H}_R$ and which have the same external address $\underline{s} = s_1 s_2 \ldots$. Let $d_k := \mathrm{Re}(z_k) - \mathrm{Re}(w_k)$ and suppose that $d_1 \geqslant 2$. Then $d_{k+1} \geqslant \exp(d_k)$ for all $k \geqslant 1$, and $z_1 = g_{\underline{s}'}(t)$ for some $\underline{s}'$ which differs from $\underline{s}$ only at finitely many entries, and $t > t_{\underline{s}} = t_{\underline{s}'}$. If also $w_1 = g_{\underline{s}'}(t')$ for some $t' > t_{\underline{s}}$, then $t > t'$.*

Proof. Let $t_k := \mathrm{Re}(w_k)$ and $u_k := \mathrm{Im}(w_k)$ for all k. Then $\mathrm{Re}(z_k) = t_k + d_k$ and $|\mathrm{Im}(z_k) - \mathrm{Im}(w_k)| < 2\pi$. Moreover,

$$|w_k| = |\lambda| \exp(\mathrm{Re}(w_{k-1})) = |\lambda| \exp(t_{k-1}),$$
$$|z_k| = |\lambda| \exp(\mathrm{Re}(z_{k-1})) = |\lambda| \exp(t_{k-1} + d_{k-1}),$$
$$|\mathrm{Im}(z_k)| \leqslant |u_k| + 2\pi \leqslant |\lambda| \exp(t_{k-1}) + 2\pi.$$

By Pythagoras' theorem, $(\mathrm{Re}(z_k))^2 = |z_k|^2 - (\mathrm{Im}(z_k))^2$, we get

$$(t_k + d_k)^2 \geqslant |\lambda|^2 (\exp(t_{k-1} + d_{k-1}))^2 - (|\lambda| \exp(t_{k-1}) + 2\pi)^2,$$

and thus

$$t_k + d_k \geqslant |\lambda| \exp(t_{k-1} + d_{k-1}) \sqrt{1 - \left(\frac{|\lambda| \exp(t_{k-1}) + 2\pi}{|\lambda| \exp(t_{k-1}) \exp(d_{k-1})} \right)^2}. \tag{9}$$

By assumption, we have $|\lambda| \exp(t_{k-1}) > |\lambda| \exp(R) > \pi$ for all k, and we take $d_{k-1} \geqslant 2$ as inductive hypothesis. Since $t_k = \mathrm{Re}(w_k) \leqslant |w_k| = |\lambda| \exp(t_{k-1})$, the fact that $\sqrt{x} \geqslant x$ for $0 \leqslant x \leqslant 1$ gives

$$\begin{aligned} d_k &\geqslant |\lambda| \exp(t_{k-1}) \left(\exp(d_{k-1}) \sqrt{1 - \left(\frac{3}{\exp(d_{k-1})} \right)^2} - 1 \right) \\ &\geqslant \pi(\exp(d_{k-1})(1 - 9\exp(-2d_{k-1})) - 1) \\ &\geqslant \pi \exp(d_{k-1}) - 9\pi \exp(-d_{k-1}) - \pi > \exp(d_{k-1}). \end{aligned}$$

This yields exponential growth of d_k and justifies the inductive hypothesis $d_{k-1} \geqslant 2$ for all k.

Choose any $\beta' < \mathrm{Re}(\kappa)$. From Equation (9), we conclude that

$$t_k + d_k + \beta' \geqslant F(t_{k-1} + d_{k-1} + \beta')$$

for all $k \geqslant k'$ depending on β', and hence, for any $N \geqslant k'$ and $k \geqslant N$,

$$t_k + d_k + \beta' \geqslant F^{\circ(k-N)}(t_N + d_N + \beta'). \tag{10}$$

Next we derive an upper bound for t_k as follows:

$$t_{k+1} = \mathrm{Re}(w_{k+1}) \leqslant |w_{k+1}| = |\lambda| \exp(t_k) = F(t_k + \mathrm{Re}(\kappa)) + 1.$$

Pick any $\beta'' > \mathrm{Re}(\kappa)$. Then, for $k \geqslant k''$ depending on β'', we have $t_{k+1} + \beta'' \leqslant F(t_k + \beta'')$. Consequently, for $N \geqslant k''$ and $k \geqslant N$, we get

$$t_k + \beta'' < F^{\circ(k-N)}(t_N + \beta''). \tag{11}$$

To find a lower bound for t_N, we use $|s_{k+1}| > F^{\circ k}(t_{\underline{s}} - \varepsilon)$ for all $\varepsilon > 0$ and infinitely many large k (depending on ε). It follows that

$$F^{\circ k}(t_{\underline{s}} - \varepsilon) < |s_{k+1}| < 2\pi(|w_{k+1}| + 1) = 2\pi(|\lambda| \exp(t_k) + 1) + 1 - 1.$$

Hence, after taking logarithms,

$$F^{\circ(k-1)}(t_{\underline{s}} - \varepsilon) < t_k + \mathrm{Re}(\kappa) + \log(2\pi) + \log\left(1 + \frac{1 + 1/2\pi}{|\lambda| \exp(t_k)}\right).$$

The logarithm is bounded above by some constant depending only on λ and R. Incorporating $\log(2\pi)$ into this constant and using equation (11), we get

$$F^{\circ(k-1)}(t_{\underline{s}} - \varepsilon) < t_k + \mathrm{Re}(\kappa) + c < F^{\circ(k-N)}(t_N + \beta'') - \beta'' + \mathrm{Re}(\kappa) + c$$

for infinitely many $k \geqslant N$.

Let $t_{\underline{s}}^{N-1} := F^{\circ(N-1)}(t_{\underline{s}})$ be the minimal potential of $\sigma^{(N-1)}(\underline{s})$. Then there is an $\varepsilon' > 0$ such that $F^{\circ(N-1)}(t_{\underline{s}} - \varepsilon) = t_{\underline{s}}^{N-1} - \varepsilon'$, and ε' tends to 0 when ε does. We get

$$F^{\circ(k-N)}(t_{\underline{s}}^{N-1} - \varepsilon') = F^{\circ(k-1)}(t_{\underline{s}} - \varepsilon) < F^{\circ(k-N)}(t_N + \beta'') - \beta'' + \mathrm{Re}(\kappa) + c$$

for infinitely many $k \geqslant N$. It follows that $t_{\underline{s}}^{N-1} - \varepsilon' \leqslant t_N + \beta''$ for any $\varepsilon' > 0$, so $t_N \geqslant t_{\underline{s}}^{N-1} - \beta''$. This yields a lower bound for t_N. Using equation (10), we get

$$t_k + d_k \geqslant F^{\circ(k-N)}(t_N + d_N + \beta') - \beta' \geqslant F^{\circ(k-N)}(t_{\underline{s}}^{N-1} - \beta'' + \beta' + d_N) - \beta'.$$

Since β' and β'' can both be arbitrarily close to $\mathrm{Re}(\kappa)$ and thus to each other, and d_N is large, it follows that there is a $t' > t_{\underline{s}}^{N-1}$ such that (with $n = k - N$) $\mathrm{Re}(E_\lambda^{\circ n}(z_N)) = t_k + d_k > F^{\circ(n-1)}(t')$ for infinitely many n. The external address of z_N is $\sigma^{(N-1)}(\underline{s})$. By Theorem 4.4, $z_N = g_{\sigma''}(t'')$ for some $\underline{s}''$ which differs from $\sigma^{N-1}(\underline{s})$ only at finitely many entries and $t'' > t_{\underline{s}}^{N-1} = t_{\underline{s}''}$. Pulling back along the orbit from z_1 to z_N, it follows that $z_1 = g_{\underline{s}'}(t)$ for some $\underline{s}'$ which differs from $\underline{s}$ only at finitely many entries, and $t = F^{\circ(-N+1)}(t'') > t_{\underline{s}} = t_{\underline{s}'}$.

If $w_1 = g_{\underline{s}'}(t')$ for $t' > t_{\underline{s}}$, then Proposition 4.5 clearly implies that $t > t'$. □

REMARK 5.2. This lemma has an interesting consequence: *if $t > t_{\underline{s}}$ is such that the orbit of $g_{\underline{s}}(t)$ is within $\mathbb{H}_{R+2}$ (with R as in the previous lemma), then all points on the entire ray segment $g_{\underline{s}}([t,\infty))$ have external address $\underline{s}$, that is, the orbit of the ray segment never crosses the partition boundary $E_\lambda^{-1}(\mathbb{R}^-)$.* If not, then by iterating finitely many times, we may assume that $g_{\underline{s}}([t,\infty))$ intersects $\mathbb{R}^-$, but the forward orbit of $g_{\underline{s}}([t,\infty))$ does not. Set $z := g_{\underline{s}}(t)$; its orbit is within $\mathbb{H}_{R+2}$, and there is a $t' > t$ such that $w := g_{\underline{s}}(t')$ has $\mathrm{Re}(w) = \mathrm{Re}(z) - 2 > R$. Now the point at greater potential has smaller real part, and this contradicts Lemma 5.1 (with a small modification if the orbit of w does not stay within $\mathbb{H}_R$).

LEMMA 5.3 (escaping points on ray). *Let $R > \log\pi - \mathrm{Re}(\kappa)$ be positive. Among all escaping points sharing any given external address $\underline{s}$ and with orbits in $\mathbb{H}_{R+2}$, there is at most one point z which is not on the dynamic ray $g_{\underline{s}}$, that is, for which there is no $t > t_{\underline{s}}$ with $z = g_{\underline{s}}(t)$.*

Proof. Suppose that there are two orbits (z_k) and (w_k) which are not on the dynamic ray $g_{\underline{s}}$ at potentials greater than $t_{\underline{s}}$. Suppose first that $|\mathrm{Re}(z_k - w_k)| < 2$ for all k. Clearly, $|\mathrm{Im}(z_k - w_k)| < 2\pi$ for all k. However, since the derivative along the orbit is uniformly bounded below by $|\lambda|e^R > \pi$, the condition $|z_k - w_k| < 2\pi + 2$ implies that $|z_{k-1} - w_{k-1}| < (2\pi + 2)/\pi$ and then $|z_{k-j} - w_{k-j}| < (2\pi + 2)/\pi^j$, so in particular, $|z_1 - w_1| < (2\pi + 2)/\pi^{k-1}$. Since k can be arbitrarily large, it follows that $z_1 = w_1$.

Therefore, we may assume that $\mathrm{Re}(z_k - w_k) \geqslant 2$ for some k. By Lemma 5.1, $z_1 = g_{\underline{s}'}(t)$ for some $\underline{s}'$ which differs from $\underline{s}$ at most at finitely many entries, and $t > t_{\underline{s}'} = t_{\underline{s}}$. The Remark 5.2 implies that $\underline{s}' = \underline{s}$. □

REMARK 5.4. Given the fact that there is an entire ray tail of escaping points with external address $\underline{s}$, and only one point with this external address (with orbit in $\mathbb{H}_R$) can be off the ray, it seems fair to say that 'almost' all escaping points are on a ray. That this is not true in the sense of dimension theory is discussed in Section 7.

The following two lemmas help to control where rays are: the first one establishes good control for certain special rays, and the second one shows that the general case is not much different if real parts are large.

LEMMA 5.5 (rays avoiding the singular strip). (1) *Let $\underline{s}$ be an external address without entry* 0. *Then the entire dynamic ray $g_{\underline{s}}((t_{\underline{s}}, \infty))$ is contained in the strip R_{s_1}.*

(2) *If, in addition,* $\mathrm{Im}(\kappa) \neq \pm\pi$, *then the closure $\overline{g_{\underline{s}}((t_{\underline{s}}, \infty))}$ is contained in R_{s_1}. The same holds if* $\mathrm{Im}(\kappa) = \pi$ *and all $s_k \notin \{0, 1\}$, or if* $\mathrm{Im}(\kappa) = -\pi$ *and all $s_k \notin \{0, -1\}$.*

Proof. By definition, every ray tail $g_{\sigma^k(\underline{s})}$ is contained in the strip $R_{s_{k+1}}$, for all $k \geqslant 0$. In particular, it is disjoint from $\mathbb{R}^-$. Therefore, $E_\lambda^{-1}(g_{\sigma(\underline{s})})$ (for any branch of E_λ^{-1}) is disjoint from all branches of $E_\lambda^{-1}(\mathbb{R}^-)$, and these are the boundaries of the strips R_j. Since the entire rays are constructed as continued pull-backs of ray tails, the first claim follows.

The assumption in the second part ensures that every ray avoids a definite neighborhood of $\mathbb{R}^-$, so in the next pull-back step, every ray avoids a definite neighborhood of all the partition boundaries, and the claim follows. □

LEMMA 5.6 (changing external addresses). *Let $\underline{s}$ and $\underline{s}'$ be two external addresses such that $|s_k - s'_k| \leqslant 1$ for every k. Then there is an $R \in \mathbb{R}$ depending only on κ with the following property: if, for $t > t_{\underline{s}}$, the orbit of $g_{\underline{s}'}(t)$ lies in $\mathbb{H}_R$, then for every $k \geqslant 0$, we have $|E_\lambda^{\circ k}(g_{\underline{s}}(t)) - E_\lambda^{\circ k}(g_{\underline{s}'}(t))| < 3\pi$.*

Proof. For k sufficiently large, this is an immediate consequence of Proposition 4.5. For the remaining smaller values of k, this follows by backwards induction: there is an $R > 0$ such that E_λ^{-1} has derivative at most 1/3 in absolute value for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > R - 3\pi$ (this is the same for every branch of E_λ^{-1}). Now, if

$$|E_\lambda^{\circ(k+1)}(g_{\underline{s}}(t)) - E_\lambda^{\circ(k+1)}(g_{\underline{s}'}(t))| < 3\pi \quad \text{and} \quad \mathrm{Re}(E_\lambda^{\circ(k+1)}(g_{\underline{s}'}(t))) > R,$$

then the same branch of E_λ^{-1} would bring them to points at distance less than π, and the branches which yield $E_\lambda^{\circ k}(g_{\underline{s}}(t))$ and $E_\lambda^{\circ k}(g_{\underline{s}'}(t))$ differ by at most $2\pi i$. □

6. *Classification of escaping points*

We will show that all escaping points are organized in the form of dynamic rays associated to exponentially bounded external addresses.

LEMMA 6.1 (escaping set connected). *Let $R > 0$ be such that $e^R \exp(\mathrm{Re}(\kappa)) \geqslant (\mathrm{Re}(\kappa) + 2\pi + 1)/(1 - 1/e)$ and suppose that (z_k) is an escaping orbit within $\mathbb{H}_R$. Then there is a closed connected unbounded set $C \subset \mathbb{H}_R$ containing z_1 such that all points of C escape within $\mathbb{H}_R$ and have the same external address as z_1. Every $z \in C$ escapes so that* $\mathrm{Re}(E_\lambda^{\circ(k-1)}(z)) \geqslant \mathrm{Re}(z_k) - \mathrm{Re}(\kappa) - 1$.

Proof. Let $\underline{s} = s_1 s_2 \ldots$ be the external address of z_1. For $k \geqslant 1$, let

$$S_k := \{z \in R_{s_k} : \mathrm{Re}(z) \geqslant \mathrm{Re}(z_k) - \mathrm{Re}(\kappa) - 1\}$$

(where R_{s_k} is the strip of points with first entry s_k in the external address). Our first claim is that $E_\lambda(S_k) \supset S_{k+1}$ for all k. Clearly $E_\lambda(S_k)$ contains all points in S_{k+1} with $|z| \geqslant (1/e)|z_{k+1}|$.

On the other hand, any $z \in S_{k+1}$ has $\mathrm{Re}(z) \geqslant \mathrm{Re}(z_{k+1}) - \mathrm{Re}(\kappa) - 1$ and

$|\mathrm{Im}(z) - \mathrm{Im}(z_{k+1})| < 2\pi$; hence $|z| \geqslant |z_{k+1}| - \mathrm{Re}(\kappa) - 2\pi - 1$. We can thus be sure that z is in $E_\lambda(S_k)$ provided that $|z_{k+1}| - \mathrm{Re}(\kappa) - 2\pi - 1 \geqslant (1/e)|z_{k+1}|$ or $|z_{k+1}| \geqslant (\mathrm{Re}(\kappa) + 2\pi + 1)/(1 - 1/e)$. However, since $|z_{k+1}| = |\lambda| \exp(\mathrm{Re}(z_k)) \geqslant |\lambda| e^R \geqslant (\mathrm{Re}(\kappa) + 2\pi + 1)/(1 - 1/e)$, the claim is proved.

Now consider, for $k \geqslant 1$, the sets

$$C_k := \{z \in S_1 : E_\lambda^{\circ i}(z) \in S_{i+1} \text{ for } i = 0, 1, \dots, k-1\}.$$

The sets $\overline{C}_k \cup \{\infty\} \subset \mathbb{P}^1$ are non-empty, compact and nested: $\overline{C}_{k+1} \subset \overline{C}_k$. We have just proved that $E_\lambda^{\circ(k-1)} : C_k \longrightarrow S_k$ is a conformal isomorphism, so all C_k are connected. The nested intersection of non-empty compact and connected sets is non-empty, compact and connected, so the set

$$C := \bigcap_{k \geqslant 1} \overline{C}_k$$

is a closed connected set with $z_1 \in C$ and $\infty \in \partial C$. For $z \in C$, we have $E_\lambda^{\circ(k-1)}(z) \in \overline{S}_k$, so $\mathrm{Re}(E_\lambda^{\circ(k-1)}(z)) \geqslant \mathrm{Re}(z_k) - \mathrm{Re}(\kappa) - 1$. This is what we claimed. □

REMARK 6.2. If we make the assumption that the orbit of z_1 never visits the sector containing 0 (so that $s_k \neq 0$ for all k), then the restriction on R is unnecessary and we can use $S_k := R_{s_k}$. In [**2**], external addresses $\underline{s}$ without entry 0 were called 'regular'. The corresponding rays are particularly easy to handle because the dynamics avoid the singular value and the real axis; see, for example, Proposition 6.11. Milnor has suggested that external addresses without entry 0 be called 'unreal'.

Lemma 5.5 shows that any two rays at 'unreal' external addresses generally have disjoint closures, so in particular they cannot land at the same point (in the sense defined below). However, in the theory of iterated polynomials it is known that most of the interest is in rays which do land together; compare with, for example, [**13**] (for instance, the entire theory of puzzles is built on such rays). Hence we try as much as possible not to restrict to 'unreal' external addresses.

The *limit set of the ray* $g_{\underline{s}}$ is defined as the set of all possible limit points of $g_{\underline{s}}(t_k)$ as $t_k \searrow t_{\underline{s}}$. We say that *the ray* $g_{\underline{s}}$ *lands at a point* w if $\lim_{t' \searrow t_{\underline{s}}} g_{\underline{s}}(t')$ exists and is equal to w (so the limit set consists of a single point). If $g_{\underline{s}}$ lands at an escaping point $w = g_{\underline{s}}(t_{\underline{s}})$, we say that *ray and landing point escape uniformly* if, for every $R \in \mathbb{R}$, there is an $N \geqslant 0$ such that for every $n \geqslant N$, we have $\mathrm{Re}(E_\lambda^{\circ n}(g_{\underline{s}}([t_{\underline{s}}, \infty)))) > R$. We show in Corollary 6.9 that whenever a ray lands at an escaping point, then the escape of ray and landing point is uniform, but this is not automatic.

LEMMA 6.3 (limit set does not intersect ray). *Suppose that a ray* $g_{\underline{s}}$ *has the property that all its points have their entire orbits in the right half plane* $\mathbb{H}_R$, *for a positive* $R > \log \pi - \log|\lambda|$. *Then* $g_{\underline{s}}$ *is disjoint from its own limit set.*

REMARK 6.4. This lemma probably holds under much weaker assumptions.

Proof of Lemma 6.3. Let L be the limit set of $g_{\underline{s}}$. Suppose that there is a $t > t_{\underline{s}}$ with $g_{\underline{s}}(t) \in L$. Pick some potential $t' \in (t_{\underline{s}}, t)$. Possibly after some finite number of iterations and using Proposition 4.5, we may assume that $\mathrm{Re}(g_{\underline{s}}(t) - g_{\underline{s}}(t')) \geqslant 3$. Then there is a potential $t'' > t_{\underline{s}}$ arbitrarily close to $t_{\underline{s}}$ so that $g_{\underline{s}}(t'')$ is arbitrarily close to $g_{\underline{s}}(t)$. More precisely, we assume that $t'' < t'$ and $\mathrm{Re}(g_{\underline{s}}(t'') - g_{\underline{s}}(t')) > 2$. However, by Lemma 5.1, this implies that $t'' > t'$, a contradiction. Hence $L \cap g_{\underline{s}}((t_{\underline{s}}, \infty)) = \emptyset$. □

We can now show that all escaping points are associated to rays.

THEOREM 6.5 (escaping points associated to ray). *Consider an exponential map for which the singular value does not escape. Then, for every escaping point* w, *there is an exponentially bounded external address* $\underline{s}$ *and a potential* $t \geqslant t_{\underline{s}}$ *such that either* $t > t_{\underline{s}}$ *and* $w = g_{\underline{s}}(t)$, *or* $t = t_{\underline{s}}$ *and the dynamic ray* $g_{\underline{s}}$ *lands at* w *with uniform escape.*

If the singular value 0 *does escape, then* $0 = g_{\underline{s}'}(t_0)$ *for some* $t_0 \geqslant t_{\underline{s}'}$, *as claimed. For any other escaping point* w, *either the claim above holds, or there is a finite* $m \geqslant 1$ *such that* $E_\lambda^{\circ m}(w) = g_{\underline{s}'}(t)$ *for* $t < t_0$.

REMARK 6.6. One can uniquely associate potentials even to those escaping points which iterate onto the same ray as the singular value, below the singular potential; however, associating an external address to them involves a non-canonical choice just as for quadratic polynomials.

Proof of Theorem 6.5. Fix $R > 0$ such that $e^R \exp(\mathrm{Re}(\kappa)) \geqslant (\mathrm{Re}(\kappa)+2\pi+1)/(1-1/e)$. Except for possibly finitely many iterations, the entire orbit of w will be in the right half plane $\mathbb{H}_R$; let us assume first that there are no exceptions at all. Then the external address of w is well defined; call it $\underline{s}$.

By Lemma 6.1, there is a closed connected unbounded set C containing w such that all points in C escape within $\mathbb{H}_R$ with external address $\underline{s}$. By Lemma 5.3, at most one point in C can fail to be on the dynamic ray $g_{\underline{s}}$. If w is on the ray at potential $t > t_{\underline{s}}$, then we are done.

Otherwise, w is the unique exceptional point in C, and every $z' \in C \backslash \{w\}$ has $z' = g_{\underline{s}}(t')$ for some $t' > t_{\underline{s}}$. We want to show that $g_{\underline{s}}((t_{\underline{s}}, \infty)) \subset C$. Suppose first that for some $t > t_{\underline{s}}$, we have $g_{\underline{s}}(t) \in C$, but not $g_{\underline{s}}([t,\infty)) \subset C$. Then $C \cup g_{\underline{s}}([t,\infty))$ disconnects $\mathbb{C}$: there is an open set $U \subset \mathbb{C}$ such that U and 0 are in different components of $\mathbb{C} \backslash (C \cup g_{\underline{s}}([t,\infty)))$. Possibly by iterating a finite number of steps, we may assume that all points in $g_{\underline{s}}([t,\infty))$ escape within $\mathbb{H}_R$; the points in C do this anyway. Moreover, for any k, all the points in $E_\lambda^{\circ k}(C)$ and $E_\lambda^{\circ k}(g_{\underline{s}}([t,\infty)))$ are in the same strip $R_{s_{k+1}}$. It follows that all points in U must escape to infinity within $\mathbb{H}_R$ by the minimum principle, and this is impossible by the classification of Fatou components (see [**6**, **7**, **8**] or [**1**]).

It follows that there is a $t^* \geqslant t_{\underline{s}}$ such that $g_{\underline{s}}((t^*,\infty)) \subset C$ but $g_{\underline{s}}(t') \notin C$ for $t_{\underline{s}} < t' < t^*$. We need to show that $t^* = t_{\underline{s}}$. If $t^* > t_{\underline{s}}$, then $g_{\underline{s}}([t^*,\infty)) \subset C$ by continuity, and $C \backslash g_{\underline{s}}([t^*,\infty))$ can contain at most one point. However, since C is closed and connected, $C = g_{\underline{s}}([t^*,\infty))$, and $w = g_{\underline{s}}(t)$ for some $t \geqslant t^* > t_{\underline{s}}$, a contradiction. We conclude that $t^* = t_{\underline{s}}$ and $g_{\underline{s}}((t_{\underline{s}},\infty)) \subset C$. Moreover, $C = g_{\underline{s}}((t_{\underline{s}},\infty)) \cup \{w\}$. Let L denote the limit set of $g_{\underline{s}}$. Since $g_{\underline{s}}((t_{\underline{s}},\infty)) \subset C$ and C is closed, we have $L \subset C = g_{\underline{s}}((t_{\underline{s}},\infty)) \cup \{w\}$. We know from Lemma 6.3 that $L \cap g_{\underline{s}}((t_{\underline{s}},\infty)) = \emptyset$, and since L is non-empty, it follows that $L = \{w\}$, so $g_{\underline{s}}$ lands at w as claimed. The escape is uniform by Lemma 6.1.

If w does not spend its entire orbit within $\mathbb{H}_R$, then there is a finite iterate which does, and which is then either on a dynamic ray or the landing point of a ray. By pulling back, the theorem then holds for w as well unless the pull-back runs through the singular value. This is never a problem for the singular value itself, and for other points the problem occurs exactly when claimed in the theorem. □

A complete classification of escaping points must describe all those external addresses $\underline{s}$ for which the ray $g_{\underline{s}}$ lands at an escaping point. To this end, we introduce the following notation.

DEFINITION 6.7 (slow and fast external addresses). We say that an external address $\underline{s}$ is *slow* if there are $A, x > 0$ and infinitely many n for which $|s_{n+k}| \leqslant AF^{\circ(k-1)}(x)$ for all $k \geqslant 1$. Otherwise, we say that $\underline{s}$ is *fast*: then, for every $A, x > 0$, all sufficiently large n have a k such that $|s_{n+k}| > AF^{\circ(k-1)}(x)$.

Clearly, any external address with $t_{\underline{s}} > 0$ or with $\liminf |s_k| = \infty$ is fast, but the converse is not true: the two external addresses 1 2 1 3 1 4 1 5 ... and 1 2 1 2 3 1 2 3 4 1 2 3 4 5 ... are both unbounded with $t_{\underline{s}} = 0$ and $\liminf |s_k| = 1$; the first one is fast, while the second one is not.

PROPOSITION 6.8 (uniform escape for fast addresses). *An external address $\underline{s}$ is fast if and only if the ray $g_{\underline{s}}$ lands at an escaping point so that ray and landing point escape uniformly (with an exception if the singular values escape so that there is an $n \geqslant 1$ and $t \geqslant t_{\underline{s}}$ with that $g_{\sigma^n(\underline{s})}(F^{\circ n}(t)) = 0$; in this case, the ray $g_{\underline{s}}$ ends at $\pm\infty$).*

Proof. Suppose that $\underline{s}$ is slow, so there are $A, x > 0$ and infinitely many n with $|s_{n+k}| \leqslant AF^{\circ(k-1)}(x)$ for all k. Then, by Proposition 3.4, for $t \geqslant x + 2\log(K+3)$, we have $g_{\sigma^n(\underline{s})}(t) = t - \kappa + 2\pi i s_{n+1} + r$ with $|r|$ universally bounded. Since $|s_{n+1}| \leqslant Ax$, it follows that $g_{\sigma^n(\underline{s})}(t)$ is bounded independently of n (for infinitely many values of n). However, $g_{\sigma^n(\underline{s})}(t) = E_\lambda^{\circ n}(g_{\underline{s}}(t_n))$ for $t_n = F^{\circ(-n)}(t)$ with $t_n \searrow 0$, so the ray cannot land at an escaping point and escape uniformly.

Conversely, suppose that $\underline{s}$ is fast. Let us first consider the special case that all entries s_k in $\underline{s}$ are different from 0, so that no ray tail $g_{\sigma^k(\underline{s})}$ is in the strip containing $\mathbb{R}^-$ and Lemma 5.5 applies (if $\mathrm{Im}(\kappa) = \pm\pi$, which is equivalent to $\lambda \in \mathbb{R}^-$, then we need to exclude both strips that are adjacent to the real axis). This case is easier because rays with zero-free external addresses respect the static partition. We then have, for any $t > t_{\underline{s}}$, the estimate

$$|E_\lambda^{\circ k}(g_{\underline{s}}(t))| \geqslant |\mathrm{Im}(E_\lambda^{\circ k}(g_{\underline{s}}(t)))| \geqslant 2\pi(s_{k+1} - 1).$$

Pick any $x > 0$, $A > 1/2\pi$ and any $t > t_{\underline{s}}$. Since $\underline{s}$ is fast, there is an N such that all $n \geqslant N$ have a k_n with $|s_{n+k_n}| > AF^{\circ(k_n-1)}(x)$. For $n \geqslant N$, let $z_n := E_\lambda^{\circ n}(g_{\underline{s}}(t))$. By Lemma 2.4, there is a $\delta > 0$ depending only on κ such that, for every $k \geqslant 0$,

$$F^{\circ k}(|z_n| + \delta) \geqslant |\mathrm{Im}(E_\lambda^{\circ k}(z_n))| = |\mathrm{Im}(E_\lambda^{\circ(n+k)}(g_{\underline{s}}(t)))| \geqslant 2\pi(s_{n+k+1} - 1).$$

Specializing k to $k_n - 1$, it follows that

$$F^{\circ(k_n-1)}(|z_n| + \delta) + 2\pi \geqslant 2\pi s_{n+k_n} \geqslant 2\pi AF^{\circ(k_n-1)}(x) \geqslant F^{\circ(k_n-1)}(x).$$

As x gets large, $|z_n|$ must also get large (if there were no additive 2π, we could conclude that $|z_n| \geqslant x - \delta$). This estimate does not depend on $t > t_{\underline{s}}$ and on $n \geqslant N$. Therefore, for every $R > 0$, there is an N such that, for all $n \geqslant N$ and $t \geqslant t_{\underline{s}}$, we have $|E_\lambda^{\circ n}(g_{\underline{s}}(t))| > R$. It follows that $\mathrm{Re}(E_\lambda^{\circ(n-1)}(g_{\underline{s}}(t))) > \log R - \mathrm{Re}(\kappa)$. If no forward iterate of the ray contains the singular value, then this implies that the ray $g_{\underline{s}}$, together with its limit set, escapes uniformly to ∞. By Lemma 6.3, the limit set of $g_{\underline{s}}$ cannot intersect the ray; since the limit set consists of escaping points, Theorem 6.5 shows that it must be a singleton; this means that the ray lands at an escaping point with uniform escape. If some forward iterate of $g_{\underline{s}}$ contains 0, then $g_{\underline{s}}$ itself ends prematurely at ∞.

For the case of a general external address $\underline{s}$, we construct a new external address $\underline{s}'$ without an entry 0 as follows: we set $s'_k := 1$ whenever $s_k = 0$, and $s'_k := s_k$ otherwise. (In the special case $\text{Im}(\kappa) = \pi$, the two strips R_0 and R_1 have $\mathbb{R}$ on their boundary and are excluded; here we set $s'_k := 2$ if $s_k = 1$ and $s'_k = -1$ if $s_k = 0$, and $s'_k := s_k$ otherwise. If $\text{Im}(\kappa) = -\pi$, we proceed likewise.) Then we have $|s_k - s'_k| \leqslant 1$ for all k, and the claim follows from Lemma 5.6: after sufficiently many iterations, the entire ray $g_{\underline{s}'}$ spends its entire orbit within any given right half plane, and then $g_{\underline{s}}$ must behave likewise (with the possible exception stated in the claim). □

COROLLARY 6.9 (classification of escaping points). *For any escaping point z, exactly one of the following three cases holds*:

(1) *There is a unique dynamic ray $g_{\underline{s}}$ and a unique $t > t_{\underline{s}}$, so that $z = g_{\underline{s}}(t)$.*

(2) *There is a unique external address $\underline{s}$ such that $g_{\underline{s}}$ lands at z, and the escape of ray and landing point is uniform.*

(3) *The singular value escapes*: $0 = g_{\underline{s}}(t)$ *for some $\underline{s}$ and $t > t_{\underline{s}}$, and the point z maps after finitely many iterations to $g_{\underline{s}}(t')$ with $t_{\underline{s}} \leqslant t' < t$.*

Rays exist exactly for exponentially bounded external addresses, and escaping landing points exist exactly for fast exponentially bounded external addresses $\underline{s}$ (unless $g_{\underline{s}'}$ lands at or contains the singular value for some $\underline{s}' = \sigma^k(\underline{s})$).

REMARK 6.10. Recall that the minimal escape potential $t_{\underline{s}} \geqslant 0$ for $\underline{s}$ has been defined in Definition 4.1, exponential boundedness of $\underline{s}$ has been defined before Lemma 2.4, and fast external addresses have been defined in Definition 6.7.

This result excludes the possibility that an escaping point is the landing point of any ray with non-uniform escape.

Proof of Corollary 6.9. We know from Theorem 6.5 that every escaping point is on a dynamic ray or the landing point of a dynamic ray with uniform escape, or (in the exceptional cases which occur only if the singular value escapes) it maps to such a point after finitely many iterations.

First we show that the three cases mentioned in the statement are mutually exclusive. All we need to show is that 'z is on a ray' and 'z is the endpoint of a ray' are mutually exclusive (we owe this argument to Lasse Rempe). To see this, fix $z = g_{\underline{s}}(t)$ with $t > t_{\underline{s}}$. We will show that there is an $\varepsilon > 0$ and a sequence $\underline{s}_n > \underline{s}$ of external addresses with $t_{\underline{s}_n} = t_{\underline{s}}$ such that $g_{\underline{s}_n}(t') \to g_{\underline{s}}(t')$ uniformly for $t' \in [t-\varepsilon, t+\varepsilon]$; there is a similar sequence $\underline{s}'_n < \underline{s}$. These two sequences of rays would intersect any dynamic ray $g_{\underline{s}'}$ which landed at z, so z cannot be the landing point of any dynamic ray. The two sequences of external addresses $\underline{s}_n$ and $\underline{s}'_n$ are easy to construct: they equal $\underline{s}$, except that the nth entry in $\underline{s}_n$ exceeds the corresponding entry in $\underline{s}$ by 1, and the nth entry in $\underline{s}'_n$ is smaller by 1. Then $E_\lambda^{\circ n}(g_{\underline{s}_n}(t')) \approx E_\lambda^{\circ n}(g_{\underline{s}}(t')) + 2\pi i$ at least for large n, and the pull-backs to $g_{\underline{s}}(t')$ and $g_{\underline{s}_n}(t')$ use the same branches of E_λ^{-1} and contract exponentially with n. Therefore, the claim follows for any $\varepsilon < t - t_{\underline{s}}$.

If z is on a dynamic ray or the landing point of $g_{\underline{s}}$ so that $g_{\underline{s}}$ and z escape uniformly, then, after some finite number of iterations, this escape takes place in $\mathbb{H}_R$ and the external address of z determines the external address of $\underline{s}$ except possibly for the first finitely many entries. By pulling back along the forward orbit of z, it follows that z is associated to only one dynamic ray which contains z or lands at z with uniform escape. Clearly the ray either contains z or lands at z.

If an escaping point is the landing point of a ray $g_{\underline{s}}$ with uniform escape and of another ray $g_{\underline{s}'}$ with non-uniform escape, then $\underline{s}$ must be fast and $\underline{s}'$ must be slow (or $\underline{s}'$ would land at another escaping point with uniform escape). Then $|s_k - s'_k|$ must be arbitrarily large, but as soon as $|s_k - s'_k| > 2$ for some iterate, the images of $g_{\underline{s}}$ and $g_{\underline{s}'}$ can no longer be injective. This is a contradiction.

The description of external addresses for rays is Theorem 4.2 and Theorem 6.5, and the classification of external addresses of escaping landing points is Proposition 6.8. □

The following result is a generalization of a result in [**2**] about *bounded* external addresses without entries 0.

PROPOSITION 6.11 (ray lands if external address has no 0). *If an external address* $\underline{s}$ *has the property that* $s_k \neq 0$ *for all* k, *then the ray* $g_{\underline{s}}$ *lands* (*in the special case when* $\mathrm{Im}(\kappa) = \pi$ *or* $\mathrm{Im}(\kappa) = -\pi$, *we have to assume that* $s_k \notin \{0, 1\}$, *respectively* $s_k \notin \{0, -1\}$).

Proof. The singular value 0 is in the strip R_0 (or in the two exceptional cases, in $R_0 \cup R_{\pm 1} \cup \mathbb{R}$). Then there is an $\varepsilon > 0$ such that every $z \in \mathbb{C}$ with $|z| < \varepsilon$ is in R_0 (respectively $R_0 \cup R_{\pm 1} \cup \mathbb{R}$).

Fix some $t > t_{\underline{s}}$ (to be adjusted below) and consider, for $k \geqslant 1$, the bounded domains

$$S_k := \{z \in R_{s_k} : \log \varepsilon - \mathrm{Re}(\kappa) < \mathrm{Re}(z) < \mathrm{Re}(E_\lambda^{\circ(k-1)}(g_{\underline{s}}(t))) + 2\}.$$

Then $E_\lambda(S_k)$ is the annulus $\varepsilon < |z| < e^2 \cdot |E_\lambda^{\circ k}(g_{\underline{s}}(t))|$ with $\mathbb{R}^-$ removed. Moreover, $E_\lambda(S_k)$ contains S_{k+1} similarly to the proof of Lemma 6.1; to see this, it suffices to check that the corners

$$c_L^{\pm} = (\log \varepsilon - \mathrm{Re}(\kappa)) + i(-\mathrm{Im}(\kappa) + 2\pi s_{k+1} \pm \pi)$$
$$c_R^{\pm} = (\mathrm{Re}(E_\lambda^{\circ(k-1)}(g_{\underline{s}}(t))) + 2) + i(-\mathrm{Im}(\kappa) + 2\pi s_{k+1} \pm \pi)$$

of S_{k+1} are in $E_\lambda(S_k)$. The outer radius of the annulus is $e^2 |E_\lambda^{\circ k}(g_{\underline{s}}(t))|$, and this is clearly greater than $\mathrm{Re}(E_\lambda^{\circ k}(g_{\underline{s}}(t))) + 2$ if t is chosen large enough. That settles $c_R^{\pm}$. For $c_L^{\pm}$, it suffices to observe that $\mathrm{Im}(c_L^{\pm}) = \mathrm{Im}(c_R^{\pm})$ and $|\mathrm{Re}(c_L^{\pm})| \leqslant |\mathrm{Re}(c_R^{\pm})|$, again provided that t is large enough. In fact, it is not hard to see that the conformal modulus of the annulus $E_\lambda(S_k) \backslash S_{k+1}$ is bounded below by some $\mu > 0$ independent of k.

Next we show that the limit set of $E_\lambda^{\circ k}(g_{\underline{s}}) = g_{\sigma^k(\underline{s})}$ must be contained in S_{k-1}: in fact, if, for some $t' > t_{\underline{s}}$, we have $\mathrm{Re}(g_{\sigma^k(\underline{s})}(t')) > \mathrm{Re}(g_{\sigma^k(\underline{s})}(t) + 2)$, then the two points $g_{\sigma^k(\underline{s})}(t')$ and $g_{\sigma^k(\underline{s})}(t)$ escape to ∞ with the same external address, and by Lemma 5.1, we would have $t' > t$. (The lemma needs the assumption that both orbits are within a certain right half plane. For $g_{\sigma^k(\underline{s})}(t)$, we can simply assume this by iterating finitely more steps if necessary; for $g_{\sigma^k(\underline{s})}(t')$, the lemma can fail only if the orbit is allowed to jump very far to the left, but we have a bound on negative real parts.)

The conclusion of the theorem is now routine: the construction ensures that the limit set is surrounded within S_1 by an infinite collection of disjoint annuli with moduli $\mu > 0$, so it must be a single point. □

REMARK 6.12. The conclusion of this theorem even holds if there are finitely many $s_k = 0$ (respectively finitely many $s_k \in \{0, \pm 1\}$), provided that the ray never maps over the singular value on its forward orbit.

REMARK 6.13. One cannot expect all rays to land. If a ray $g_{\underline{s}}$ does not land (and does not hit the singular value during its orbit), then its external address $\underline{s}$ must be slow, and it must contain entries 0 (even infinitely many of them, or it would be the pull-back of a ray which lands). We believe that the limit set of such a ray cannot contain escaping points, but we cannot prove this (any such limiting escaping point is of course covered by our classification, so it must be part of some ray).

7. *Dimension and escape*

In this section, we give several conclusions which follow from our classification and our estimates: we generalize the Karpińska paradox, and we show that all points on rays 'zip to infinity' in the sense of Rippon and Stallard.

COROLLARY 7.1 (the dimension paradox). *The union of all dynamic rays has Hausdorff dimension* 1, *while the set of escaping ray endpoints has Hausdorff dimension* 2.

Proof. In Proposition 4.5, we proved that every point z on a ray satisfies, for every $p > 0$, the parabola condition $|\mathrm{Im}(E_\lambda^{\circ n}(z))|^p < \mathrm{Re}(E_\lambda^{\circ n}(z))$ for all but finitely many n. Karpińska [**9**] proved that the set of escaping points which satisfies the parabola condition for p has Hausdorff dimension at most $1 + 1/p$. This proves the first claim.

On the other hand, McMullen [**11**] proved that the entire set of escaping points has Hausdorff dimension 2. Our classification shows that any escaping point which is not on a ray is the endpoint of a unique ray, and this settles the second claim. □

This result has been shown by Karpińska for real parameters $\lambda \in (0, 1/e)$ using properties of the Julia set related to the existence of an attracting fixed point. Since we prove in Proposition 4.5 a stronger escape condition than the parabola condition, one can probably improve the estimate of the Hausdorff dimension, perhaps leading to a gauge function like $t/(\log(1/t))^n$ for some n.

It is known from [**7**] that the set of escaping points has measure zero. (Note, however, that for the map $\exp(z)$, the singular value escapes, and almost every orbit is asymptotic to the postsingular set [**10**]: a typical orbit follows the singular orbit for a while, and then returns far into a left half plane and follows the singular orbit for a longer time.) Our methods and classification should apply to larger classes of functions for which the escaping set has positive measure (such as the family $z \longmapsto ae^z + be^{-z}$ [**11**]). This is currently work in progress. (In fact, recent work by Günter Rottenfußer and the first author [**21**] confirms this expectation for maps $ae^z + be^{-z}$: the classification of escaping points for such maps is essentially the same, and the union of all dynamic rays still has dimension 1, while the escaping endpoints form a set of positive planar Lebesgue measure, and in certain cases even of full measure.)

Rippon and Stallard, in a recent paper [**15**], consider the set of escaping points

$I(f)$ of a transcendental meromorphic function f and the subsets

$$I'(f) := \left\{ z \in I(f) : \frac{\ln |f^{\circ(n+1)}(z)|}{\ln |f^{\circ n}(z)|} \to \infty \text{ as } n \to \infty \right\}$$

$$Z(f) :=: \left\{ z \in I(f) : \frac{\ln \ln |f^{\circ n}|}{n} \to \infty \text{ as } n \to \infty \right\}.$$

They call $Z(f)$ the set of points which 'zip to infinity', and they show various properties of $I'(f)$ and $Z(f)$ by analogy to known properties of $I(f)$, such as the fact that the boundary of any of these sets equals the Julia set of f. For the restricted family of our maps, our estimates lead to a good description of $I'(f)$ and $Z(f)$ because all rays are contained in $I'(f) \cap Z(f)$.

COROLLARY 7.2 (points on rays zip to infinity). *Every dynamic ray is entirely contained in the sets $I'(f)$ and $Z(f)$, and $\overline{I'(f)} = \overline{Z(f)} = \overline{I(f)} = J(f)$, where $J(f)$ is the Julia set of $f = E_\lambda$.*

Proof. The estimates in Proposition 4.5 show immediately that for any external address $\underline{s}$ and any potential $t > t_{\underline{s}}$, we have $z = g_{\underline{s}}(t) \in I'(f)$ and $z = g_{\underline{s}}(t) \in Z(f)$. We also know that every escaping point is in the closure of a single ray, so $\overline{I'(f)} = \overline{Z(f)} = \overline{I(f)}$. It is well known that $\overline{I(f)} = J(f)$ for entire maps with finitely many singular values (this is a special case of [**5**, p. 344]). For our maps, we noted earlier that $I(f) \subset J(f)$, and since the Julia set is closed, we have $\overline{I(f)} \subset J(f)$. Conversely, it follows immediately from Montel's theorem that escaping points are dense in the Julia set. □

Acknowledgements. This project was inspired by discussions with Bogusia Karpińska and Misha Lyubich at a Euroconference in Crete organized by Shaun Bullett, Adrien Douady and Christos Kourouniotis. We also thank Bob Devaney, Núria Fagella, John Hubbard and Lasse Rempe for interesting discussions. We gratefully acknowledge support and encouragement by John Milnor and the Institute for Mathematical Sciences in Stony Brook. Much of this work was carried out while we held positions at the Ludwig-Maximilians-Universität München and the Technische Universität München, respectively.

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