On Fibers and Local Connectivity of Mandelbrot and Multibrot Sets

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Abstract. We give new proofs that the Mandelbrot set is locally connected at every Misiurewicz point and at every point on the boundary of a hyperbolic component. The idea is to show "shrinking of puzzle pieces" without using specific puzzles. Instead, we introduce *fibers* of the Mandelbrot set (see Definition 4.2) and show that fibers of certain points are "trivial", i.e., they consist of single points. This implies local connectivity at these points.

Locally, triviality of fibers is strictly stronger than local connectivity. Local connectivity proofs in holomorphic dynamics often actually yield that fibers are trivial, and this extra knowledge is sometimes useful. We include a proof that local connectivity of the Mandelbrot set implies density of hyperbolicity in the space of quadratic polynomials (Corollary 4.6).

We write our proofs more generally for the *"Multibrot sets"* $\mathcal{M}_d := \{c \in \mathbb{C} \colon \text{the Julia set of } z \mapsto z^d + c \text{ is connected}\}$.

Contents

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1. Introduction

A great deal of work in holomorphic dynamics has been done in recent years trying to prove local connectivity of Julia sets and at many points of the Mandelbrot set $\mathcal{M}$, notably by Yoccoz, Lyubich, Levin, van Strien, Petersen and others. One reason for this work is that the *topology of Julia sets and the Mandelbrot set* is completely described once local connectivity is known: if such a set is locally connected, then it is homeomorphic to quite a simple combinatorial-topological "pinched disk" model [**Do**]. Another reason is that local connectivity of $\mathcal{M}$ implies that *hyperbolicity is dense (and of course open)* in the space of quadratic polynomials [**DH1**]. By universality of the Mandelbrot set [**DH2, McM**], this would imply that hyperbolicity was open and dense in many other one-dimensional spaces of rational and even transcendental maps; see for example [**F**] for the family of cubic polynomials with a superattracting fixed point: local connectivity of the bifurcation locus can be established at every point which is not on the boundary of an embedded copy of the Mandelbrot set, and a non-hyperbolic open set in parameter space would be contained within a homeomorphic copy of $\mathcal{M}$. A third reason is that local connectivity of $\mathcal{M}$ implies *combinatorial rigidity* of quadratic polynomials: if two quadratic polynomials in $\mathcal{M}$ are not conformally conjugate to each other, then they can be distinguished already by the combinatorics of which periodic dynamic rays land at the same points (provided that all periodic orbits are repelling). Douady has coined the term "points are points" for this property: a "combinatorial point" in $\partial\mathcal{M}$ is the set of parameters with combinatorially equivalent dynamics, and the task is to show that every combinatorial point is a single point in $\mathbb{C}$.

Landing properties of external rays of a compact connected and full set $K \subset \mathbb{C}$ (such as the Mandelbrot set and Julia sets) are closely related to local connectivity of K: by Carathéodory's theorem [**M1**, Theorem 17.14], the set K is locally connected if and only if all external rays *land* such that the landing points depend continuously on the angle. This yields a continuous surjection $\mathbb{S}^1 \to \partial K$ known as the *Carathéodory loop* (each angle $\vartheta \in \mathbb{S}^1$ maps to the landing point of the ray at angle ϑ). Even if K is not known to be locally connected, most external rays land for general reasons ([**M1**, Theorem 17.4] or [**P**, Theorem 1.7]) (although in general one cannot easily tell whether or not a given ray lands). If the set K is locally connected at some $z \in \partial K$, then it "almost" follows that z is the landing point of at least one external ray: local connectivity at z is not quite good enough (nor even a necessary condition); but many proofs of local connectivity show a slightly stronger property which does imply that z is the landing point of external rays (see below).

The boundary behavior of Riemann maps, and the landing of external rays, is studied by the classical theory of prime ends [**P, M1**]. This includes the concept of impressions defined in Section 2. In complex dynamics, it has often proved useful to approach the boundary in more than one direction at a time, obtaining information for example from the fact that several external rays land at the same point. A fundamental construction in many proofs of local connectivity is the *puzzle* technique introduced by Branner, Hubbard and Yoccoz [**BH, H, M2**]. Local connectivity at a point $z \in \partial K$ is established by proving *shrinking of puzzle pieces* around z; if these puzzle pieces shrink to $\{z\}$, then K is locally connected at z and external rays land at z.

At present, local connectivity of many Julia sets is known, while local connectivity of the Mandelbrot set is still a conjecture — arguably the principal conjecture in the field. However, it is known that $\mathcal{M}$ is locally connected at many of its boundary points [**H, L**]. This was proved using puzzle techniques, which implies that external rays of $\mathcal{M}$ land at these points.

The present paper has several purposes: one of them is to give a condition "slightly stronger" than local connectivity which implies that a point $z \in \partial\mathcal{M}$ is the landing point of an external ray. We introduce *fibers* of $\mathcal{M}$ as the collection of points in $\mathcal{M}$ which will always be in the same puzzle piece, no matter how the puzzle is constructed. Our arguments will thus never use specific puzzles. We say that the fiber of some $c \in \mathcal{M}$ is *trivial* if it consists of c alone. If $c \in \partial\mathcal{M}$ has trivial fiber, then it follows that $\mathcal{M}$ is locally connected at c and that c is the landing point of external rays. Douady's joke "points are points" thus acquires the more precise meaning "fibers are points".

It turns out that the idea of fibers allows to give easy proofs of certain fundamental and classical results about local connectivity of $\mathcal{M}$: we prove that all boundary points of hyperbolic components and all Misiurewicz points have trivial fibers.

One problem in holomorphic dynamics is that many results are folklore, with few accessible proofs published. In particular, many of the fundamental results about the Mandelbrot set, due to Douady and Hubbard, have been described in their famous "Orsay notes" [**DH1**], which have never been published. They are no longer available, and it is not always easy to pinpoint a precise reference even within these notes. Meanwhile, many of these results have been proved or at least sketched in [**H, M3, S1, PR, TL2, Do**]. This paper provides proofs, some of them new, of certain key results about the Mandelbrot set, such as the fact that local connectivity implies density of hyperbolicity in the space of quadratic polynomials. At the end, we discuss some of the consequences of local connectivity of $\mathcal{M}$ as mentioned above: we relate combinatorial classes to fibers, and we briefly mention the pinched disk model of $\mathcal{M}$.

While our main results on the Mandelbrot set are known, the proofs are more combinatorial-topological and less complex analytic than known proofs, and they thus apply in other circumstances (such as for the "Tricorn"). We write our proofs more generally for parameter spaces of polynomials of the form $z \mapsto z^d + c$. In addition, fibers have pleasant built-in properties related, for instance, to renormalization and tuning [**S4**]: a point $c \in \mathcal{M}$ has trivial fiber if and only if the point $\Psi(c)$ has trivial fiber, for every "tuned copy" $\Psi\colon \mathcal{M} \to \mathcal{M}$ of the Mandelbrot set within itself. Similar results hold for other partial homeomorphisms of parts of $\mathcal{M}$ into itself, such as those obtained by quasiconformal surgery.

We discuss polynomials of the form $p_c\colon z \mapsto z^d + c$, for arbitrary complex constants c and arbitrary degrees $d \geq 2$. These are, up to normalization, exactly those polynomials which have a single critical point. Following a suggestion of Milnor, we call these polynomials *unicritical* (or unisingular). We will always assume unicritical polynomials to be normalized as above, and the variable d will always denote the degree. We define the *Multibrot set* of degree d as the connectedness locus of these families, that is

$$\mathcal{M}_d := \{c \in \mathbb{C} : \text{ the Julia set of } z \mapsto z^d + c \text{ is connected } \} .$$

In the special case $d = 2$, we obtain quadratic polynomials, and $\mathcal{M}_2 = \mathcal{M}$ is the familiar *Mandelbrot set*. All the Multibrot sets are connected, they are symmetric with respect to the real axis, and they also have $d-1$-fold rotation symmetries (see [**LS2**] with pictures of several of these sets). The present paper can be read with the quadratic case in mind throughout. However, we have chosen to do the discussion for all the Multibrot sets because this requires only occasional slight modifications – and because recently interest in the higher degree case has increased; see e.g. Levin and van Strien [**LvS**] or McMullen [**McM**].

In a related paper [**S2**], we have introduced the concept of fibers for arbitrary compact connected and full subsets of $\mathbb{C}$, and we have applied it to Julia sets. We allowed for some overlap with [**S2**] in order to make the present paper more self-contained: Fibers of Mandelbrot and Multibrot sets are born with nicer properties than can be expected in general, and this simplifies the discussion. An earlier version of this paper appeared as [**S3**].

We begin in Section 2 by a review of certain important properties of Mandelbrot and Multibrot sets. The main result in Section 3 is the "Branch Theorem" which states that branch points (in a certain sense) in Multibrot sets are postcritically finite. We then define fibers for these sets in Section 4. We also show that the fiber of an interior point is trivial if and only if it is in a hyperbolic component. We conclude the section by a proof that local connectivity *of the entire set* $\mathcal{M}_d$ is equivalent to triviality of *all* fibers, and both conditions imply density of hyperbolicity (using an argument of Douady and Hubbard; Corollary 4.6). Note however that in general, triviality of the fiber of some $c \in \mathcal{M}_d$ is strictly stronger than local connectivity of $\mathcal{M}_d$ at c.

We then prove that every Multibrot set has trivial fibers and is thus locally connected at every boundary point of a hyperbolic component and at every Misiurewicz point. Boundaries of hyperbolic components are discussed in Section 5, except roots of primitive components: they require special treatment which can be found in Section 6. In Section 7 we prove that fibers of Misiurewicz points are trivial. This shows that fibers of Multibrot sets have particularly convenient properties. Finally, in Section 8, we compare fibers to combinatorial classes.

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2. Multibrot Sets

In this section, we review some necessary background about Multibrot sets and prove several fundamental properties. Most of these are originally due to Douady and Hubbard [**DH1**] in the quadratic case; more recent references include [**M3, S1**] and, for degrees $d \geq 2$, [**Eb, ES, PR**]. All Multibrot sets $\mathcal{M}_d$ are compact, connected and full (a compact subset of $\mathbb{C}$ is called *full* if its complement in $\mathbb{C}$ is connected). It is conjectured but not yet known that the Multibrot sets are locally connected. However, it is known that many of its fibers are trivial. We will show this for certain particularly important fibers in Sections 5, 6 and 7. We use the standard notations $\overline{\mathbb{C}}$ for the Riemann sphere and $\mathbb{D}$ for the open unit disk in $\mathbb{C}$.

• Parameter Rays, Dynamic Rays, and Ray Pairs

We recall the definition of external rays: for a compact connected and full subset $K \subset \mathbb{C}$ consisting of more than a single point, there is a unique conformal isomorphism $\Phi\colon \overline{\mathbb{C}} \setminus K \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ fixing ∞, normalized so that $\lim_{c\to\infty} \Phi(c)/c \in \mathbb{R}^+$. Inverse images of radial lines in $\mathbb{C} \setminus \overline{\mathbb{D}}$ are called *external rays*, and an external ray at some angle ϑ is said to *land* if the limit $\lim_{r\searrow 1} \Phi^{-1}(re^{2\pi i\vartheta})$ exists. The *impression* of this external ray is the set of all limit points of $\Phi^{-1}(r'e^{2\pi i\vartheta'})$ for $r' \searrow 1$ and $\vartheta' \to \vartheta$. Note that we measure external angles in full turns: they live in $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$.

As in [**S1**] and [**M3**], we will call external rays of Multibrot sets *parameter rays* in order to distinguish them from external rays of Julia sets, which are called *dynamic rays*. The parameter ray at angle ϑ for some $\mathcal{M}_d$ will be denoted $R_{\mathcal{M}_d}(\vartheta)$ or simply $R_{\mathcal{M}}(\vartheta)$, while the dynamic ray at angle ϑ for the Julia set at parameter c will be denoted by $R_c(\vartheta)$ or simply $R(\vartheta)$. All parameter rays at rational angles are known to land (see Douady and Hubbard [**DH1**], Schleicher [**S1**], Petersen and Ryd [**PR**], Eberlein [**Eb, ES**] or, in the periodic case, Milnor [**M3**]).

A *ray pair* is a collection of two external rays (dynamic or parameter rays) which land at a common point. If a ray pair is periodic or preperiodic (i.e. both rays are periodic resp. preperiodic), then the two rays have equal periods and preperiods. A ray pair at angles (α, α') *separates* two points $z, z' \in \mathbb{C}$ if the two points are in different connected components of $\mathbb{C} \setminus \overline{R(\alpha) \cup R(\alpha')}$ (in particular, a ray pair does not separate its landing point from any other point).

A dynamic ray pair is *characteristic* if it separates the critical value from the critical point and from all the other rays on the forward orbit of the ray pair. A periodic or preperiodic point is characteristic if it is the landing point of a characteristic ray pair. If a periodic ray pair is not characteristic, then exactly one of the finitely many ray pairs on the forward orbit of the ray pair is characteristic [**M3**, Lemma 2.11]. Every preperiodic ray pair has exactly one periodic characteristic ray pair on its forward orbit and possibly one or several preperiodic ones. The landing point of a periodic or preperiodic dynamic ray pair is always on a repelling or parabolic orbit; if a preperiodic dynamic ray pair is characteristic, then its landing point is necessarily on a repelling orbit.

The following theorem will be used throughout this paper. The first half is due to Lavaurs [**La**] for degree $d = 2$; the arguments in [**M3**] generalize to prove the full statement.

Theorem 2.1 (Correspondence of Ray Pairs).
For every degree $d \geq 2$ and every unicritical polynomial $z \mapsto z^d + c$ with $c \in \mathcal{M}_d$, there are bijections between the parameter ray pairs at periodic and preperiodic angles which separate 0 and c, and the characteristic periodic and preperiodic ray pairs in the dynamic plane of c landing at repelling orbits. This bijection preserves external angles.

The critical value is never the landing point of a periodic dynamic ray. It is the landing point of a preperiodic dynamic ray if and only if the critical orbit is strictly preperiodic; in this case, the external angles of the parameter rays landing at c (the parameter) are the same as the external angles of the dynamic rays landing at c (the critical value). □

Not all periodic and preperiodic parameter rays are organized in pairs. The number of parameter rays at preperiodic angles landing at a common point can be any positive integer. For parameter rays at periodic angles, this number is either 1 or 2; in the quadratic case, this number is always 2 (we count the parameter rays at angles 0 and 1 separately).

All parameter rays at periodic angles are known to land at parabolic parameters (those parameters for which the critical orbit converges to a unique parabolic orbit). All parameter rays at preperiodic angles land at parameters where the critical value is strictly preperiodic: such parameters are (somewhat unfortunately) known as "Misiurewicz points".

• Hyperbolic Components

A *hyperbolic component* of $\mathcal{M}_d$ is a connected component of the set $\{c \in \mathcal{M}_d \colon z^d + c \text{ has an attracting periodic orbit}\}$. The results in this subsection go back to [**DH1**] for $d = 2$; see also [**M3, S1**] and, for $d \geq 2$, [**Eb, ES**].

Theorem 2.2 (Hyperbolic Components). *Every hyperbolic component W is open, and every $c \in W$ has a unique attracting periodic orbit, the period of which is constant throughout W. The multiplier induces a holomorphic map $\mu \colon W \to \mathbb{D}$ which extends continuously to $\mu \colon \overline{W} \to \overline{\mathbb{D}}$ as a branched cover of degree $d-1$, ramified only over the value* 0. □

In particular, μ restricts to a degree $d-1$ cover $\partial W \to \partial\mathbb{D}$. We say that $c \in \partial W$ has *internal angle* ϑ if $\mu(c) = e^{2\pi i \vartheta}$.

Every hyperbolic component W has a unique *center*: this is the unique $c \in W$ with $\mu(c) = 0$; in the corresponding dynamics, the

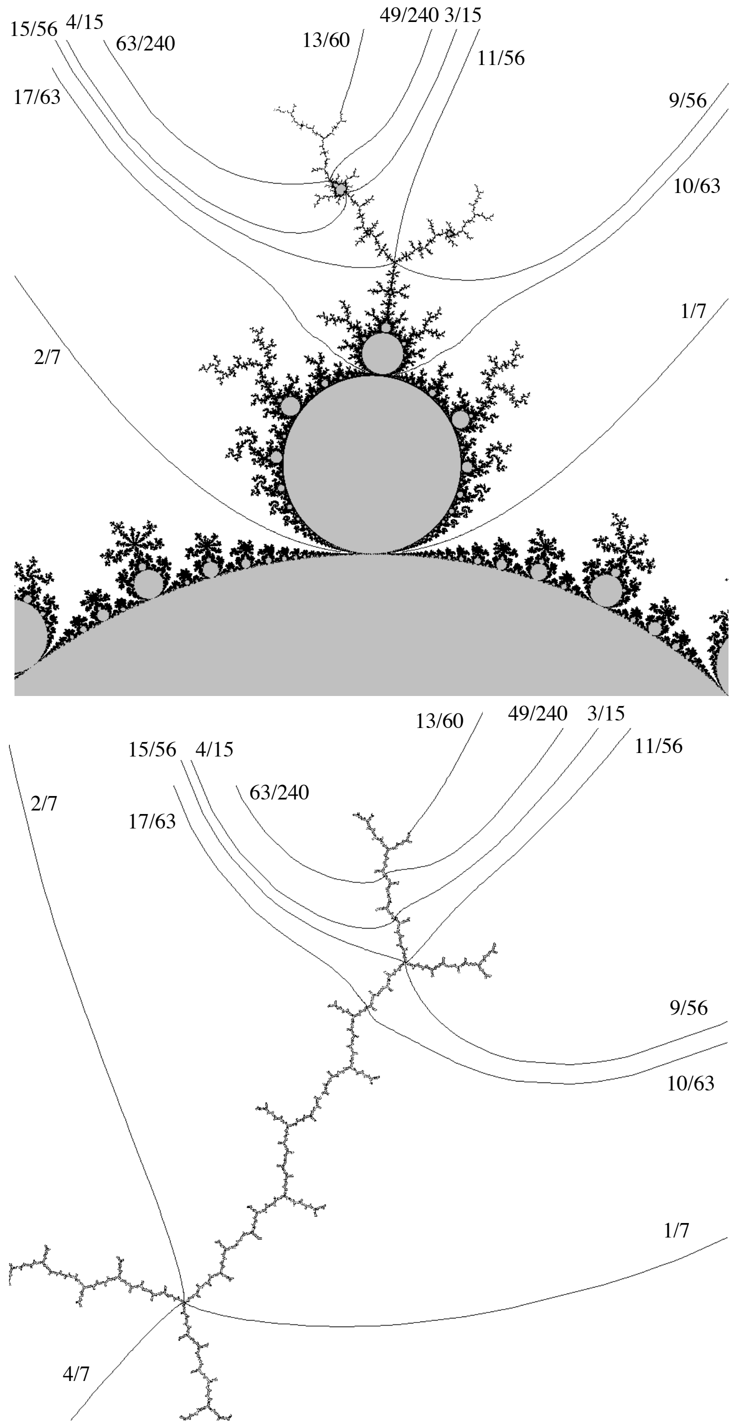

FIGURE 1. *Several parameter and dynamic ray pairs illustrating Theorem 2.1; see next page.*

attracting orbit is superattracting, so the critical orbit is periodic. If the period of W is at least 2, then there is a unique parameter $c_W \in \partial W$ which is the landing point of two parameter rays which separate W from 0. The parameter c_W is called the *root* of W. The multiplier map $\mu\colon \overline{W} \to \overline{\mathbb{D}}$ always has $\mu(c_W) = 1$. There are $d-2$ further parameters $c \in \partial W$ with $\mu(c) = 1$ but $c \neq c_W$; such parameters are called *co-roots* of W. While each root is the landing point of exactly two parameter rays, every co-root is the landing point of exactly one parameter ray. All these parameter rays have periodic angles with the same periods as the hyperbolic components at whose (co-)roots the rays are landing. Dynamically, roots and co-roots are parameters with parabolic orbits the period of which divides the period of W. At the root, every parabolic periodic point is the landing point of at least two periodic dynamic rays, while at co-roots, exactly one periodic dynamic ray lands at each parabolic periodic point. For period 1 and degrees $d > 2$, the $d-1$ boundary parameters of W with $\mu(W) = 1$ have conjugate dynamics, and it makes little sense to distinguish between root and co-roots.

In the example in Figure 1, the parameter rays at angles $1/7$ and $2/7$ land at the root of a hyperbolic component of period 3, and the rays at angles $10/63$ and $17/63$ land at the root of a component of period 6. Both components are above the parameter ray pairs landing at their roots.

Every parabolic parameter is root or co-root of a unique hyperbolic component, and every periodic parameter ray lands at the root or at a co-root of a hyperbolic component (of equal period).

FIGURE 1. (ON PREVIOUS PAGE) *Top: Detail of the Mandelbrot set with several parameter ray pairs at periodic and preperiodic angles drawn in, together with the parameter ray at angle* $13/60$ *which lands alone at some parameter* $c_{13/60} \in \mathcal{M}_2$ *and which is separated from the origin by all ray pairs shown. Bottom: The dynamical plane for the parameter* $c_{13/60}$ *(detail), with the dynamic rays at the same angles drawn in. The dynamic ray at angle* $13/60$ *lands at the critical value, the others form the same ray pairs as in parameter space, and they all separate the critical value from the origin. Note that the rays at angles* $1/7$ *and* $2/7$ *form a characteristic ray pair; the ray at angle* $4/7$ *lands at the same point, but is not part of a characteristic ray pair.*

No co-root is on the boundary of another hyperbolic component. If the root of a hyperbolic component W is on the boundary of another hyperbolic component W', then the period of W' strictly divides that of W, and W is called a *satellite* component (or *non-primitive*); otherwise, W is called *primitive*. If two hyperbolic components have a common boundary point, then this common boundary point is the root of one component, which is a satellite of the other. If ϑ and ϑ' are the two external angles of the parameter rays landing at the root of W, then W is a satellite if and only if $d^k\vartheta = \vartheta'$ and $d^{k'}\vartheta' = \vartheta$ for some integers $k, k' > 0$; otherwise, W is primitive and $d^k\vartheta \neq \vartheta'$, $d^{k'}\vartheta' \neq \vartheta$ for all $k, k' > 0$.

For every hyperbolic component W of $\mathcal{M}_d$ with period n, every $c \in \partial W$ with $\mu(c) = e^{2\pi i p/q}$ and $p/q \in (\mathbb{Q} \setminus \mathbb{Z})/\mathbb{Z}$ in lowest terms is the root of a satellite component $W_{p/q}$ of period qn. This component $W_{p/q}$ is sometimes called a p/q-satellite of W.

• Wakes and Limbs

The *wake* of a hyperbolic component W is the connected open domain in the complex plane separated from the origin by the two periodic parameter rays landing at the root of W. A *p/q-subwake* of W is the wake of a p/q-satellite component of W. The intersection of $\mathcal{M}_d$ with the wake or a p/q-subwake of W is called the *limb* or *p/q-sublimb* of W. For a hyperbolic component W with root c_W and limb L_W, the set $L_W \cup \{c_W\}$ is compact. It is the root c_W which disconnects L_W from the rest of $\mathcal{M}_d$; we will show in Corollary 5.2 that each limb is connected.

All the finitely many preperiodic parameter rays landing at any Misiurewicz point cut $\mathbb{C}$ into as many open parts as there are rays. The part containing the origin will be called the *wake exterior* of the Misiurewicz point, while all the other parts are its *subwakes*. The union of all subwakes, together with the parameter rays between them, will be called the *wake* of the Misiurewicz point: this is the complement in $\mathbb{C}$ of the closure of the zero subwake. If only one ray lands at the Misiurewicz point, then the wake is empty, and the wake exterior is the entire complex plane minus the ray and its landing point.

Centers of hyperbolic components and Misiurewicz points together form the countable set of *postcritically finite* parameters. When treating them simultaneously, it will sometimes simplify language to identify a hyperbolic component with its center and speak e.g. of the "wake of c_0", meaning of course the wake of the component with center c_0.

The following result was originally proved by Yoccoz for degree $d = 2$ using a bound on sizes of sublimbs (the "Yoccoz inequality"); see Hubbard [**H**, Section I.4]. That proof, as well as ours, uses in an essential way the following fact due to Douady: every repelling periodic point in a connected Julia set is the landing point of at least one periodic dynamic ray [**H**, Theorem I.A], [**M1**, Theorem 18.11].

THEOREM 2.3 (No Irrational Subwakes).
Any point in $\mathcal{M}_d$ within the wake of a hyperbolic component is either in the closure of the component or within one of its sublimbs at rational internal angles with denominator at least two.

REMARK. Sometimes, this theorem is phrased as saying that hyperbolic components have "no ghost limbs": for every hyperbolic component, decorations within the component's wake are attached only at parabolic boundary points, but not at the root, at co-roots, or at irrational boundary points.

PROOF. Let W be a hyperbolic component, let n be its period and let $\tilde{c}$ be a point within the limb of the component but not on the closure of W. There are finitely many hyperbolic components of periods up to n, some of which might possibly be within the wake of W. We lose nothing if we assume that $\tilde{c}$ is outside the closures of the wakes of such hyperbolic components, possibly after replacing $\tilde{c}$ with a different parameter: if there was a hyperbolic component of period up to n in an "irrational sublimb" of W, then this same "irrational sublimb" is connected and thus contains points arbitrarily close to W. More precisely, we can argue as follows: the rational parameter rays landing on the boundary of W split into two groups according to whether they pass $\tilde{c}$ to the "left" or "right", and the region in the wake of W sandwiched between these rays is connected. (Moreover, it is quite easy to show directly that every hyperbolic component in the wake of W is contained in a subwake at rational internal angle, for example using Hubbard trees or a formula for the "width" of subwakes.)

Denote the center of W by c_0. There is a path γ connecting c_0 to $\tilde{c}$ within the wake of W and avoiding the closures of the wakes of all hyperbolic components of periods up to n within the wake of W, except W itself. This path need not be contained within $\mathcal{M}_d$. For the parameter c_0, the critical value is a superattracting periodic point. We want to continue this periodic point analytically along γ, obtaining an analytic function $z(c)$ such that the point $z(c)$ is periodic for the parameter c on γ. This analytic continuation is uniquely possible because we never encounter multipliers $+1$ for a period-n orbit along γ. Therefore, we obtain a unique periodic point $z(\tilde{c})$ which is repelling. Since $\tilde{c}$ is within the connectedness locus, the point $z(\tilde{c})$ is the landing point of finitely many dynamic rays at periodic angles.

For the parameter c_0, all the periodic dynamic rays of periods up to n land at repelling periodic points of periods up to n. These periodic points can all be continued analytically along γ, they remain repelling and keep their dynamic rays because the curve avoided parameter rays of periods up to n which make up wake boundaries. Therefore, at the parameter $\tilde{c}$, there is no dynamic ray of period n available to land at $z(\tilde{c})$, so that the rays landing at this point must have periods sn, for some $s \geq 2$. Therefore, at least s rays must land at $z(\tilde{c})$ (in fact, the number of rays must be exactly s). Now let $\tilde{U}$ be the largest open neighborhood of $\tilde{c}$ in which this periodic orbit can be continued analytically as a repelling periodic point, retaining all its s dynamic rays. This neighborhood is the *wake of the orbit* and is at the heart of Milnor's discussion in [**M3**]. This wake is bounded by two parameter rays at periodic angles, landing together at a parabolic parameter of ray period sn and orbit period n. Denote this landing point by $\tilde{c}_0$. Obviously, $\tilde{U}$ cannot contain the hyperbolic component W, so $\tilde{U}$ must be contained within the wake of W.

The point $\tilde{c}_0$ is on the boundary of a hyperbolic component of period at most n within the wake of W. The wake of this component contains $\tilde{U}$ and thus $\tilde{c}$, so this component can only be W by the assumptions on $\tilde{c}$. It follows that $\tilde{U}$ is a subwake of W at a rational internal angle. □

Remark. This result goes a long way towards proving local connectivity of the Multibrot sets at boundary points of hyperbolic components, as was pointed out to us by G. Levin: see Corollary 5.1.

3. Hubbard Trees and Branch Points

In this section, we introduce *Hubbard trees* as dynamically significant subsets of Julia sets with a number of useful properties. We exploit this information in order to get a tree-like structure for the Multibrot sets, at least from a combinatorial point of view. The most important fact is the Branch Theorem 3.1 which asserts that branch points in Multibrot sets are postcritically finite.

• Hubbard Trees

In [**DH1**], Hubbard trees were introduced for arbitrary postcritically finite polynomials; we need them only for Misiurewicz parameters c. In this case, the Julia set J of c is a *dendrite:* it is compact, connected, locally connected and has no interior, and its complement is connected [**M1**, Theorem 19.7]. Therefore, J is uniquely arcwise connected: any two points $z, z' \in J$ are connected by an arc $[z, z'] \in J$ which is unique up to reparametrization. To see this, recall first that compact connected locally connected metric spaces are arcwise connected [**M1**, Lemmas 17.17 and 17.18]; if there were two arcs in J connecting z and z' along different subsets of J, then the union of these two arcs would disconnect $\mathbb{C}$. Clearly, J has d-fold rotational symmetry around 0, and if $\zeta \neq 1$ is a d-th root of unity, then $0 \in [z, \zeta z]$ for any $z \in J$.

In the Misiurewicz case, the critical orbit consists of finitely many points $c_0 = 0, c_1 = c, c_2, \ldots \in J$ with $c_{k+l} = c_l$ for some $k, l > 0$; choosing k and l minimal, it follows that $c_{k+l-1} = \zeta c_{l-1}$, where $\zeta \neq 1$ is a d-th root of unity. Moreover, $l \geq 2$ because the critical value $c_1 = c$ has only $c_0 = 0$ as preimage. The union of the finitely many arcs $[c_i, c_j] \subset J$ forms a finite topological tree within J, called the *Hubbard tree* T. By construction, all endpoints of T are on the critical orbit. Since $c_{k+l-1} = \zeta c_{l-1}$, it follows that $0 \in [c_{k+l-1}, c_{l-1}]$, so 0 is not an endpoint of T.

For all $c_i, c_j \in T$, we have $[p_c(c_i), p_c(c_j)] = [c_{i+1}, c_{j+1}] \in T$. Therefore, $p_c([c_i, c_j]) = [c_{i+1}, c_{j+1}]$ if $0 \notin [c_i, c_j]$, while $p_c([c_i, c_j]) = [c_{i+1}, c] \cup [c, c_{j+1}]$ otherwise. Since T is the union of arcs $[c_i, c_j]$, it follows that $p_c(T) \subset T$. In fact, $p_c(T) = T$: this is because $p_c(T)$ is clearly a connected subset of T containing $c_1, c_2, \ldots$, hence all endpoints of T.

If $z \in T$ is not an endpoint, then $p_{c_0}(z)$ is not an endpoint either unless $z = 0$ (because p_c is a local homeomorphism near every $z \neq 0$). If the critical value c_1 was not an endpoint, then no point on the critical orbit could be an endpoint, which is a contradiction to the fact that all endpoints of T are on the critical orbit. Therefore, c_1 is always an endpoint.

The p_c-image of any branch point $z \neq 0$ is another branch point with at least as many edges (possibly more). Since T is a finite tree with finite critical orbit, it follows that all branch points are periodic or preperiodic, and all the periodic branch points have the same number of incident edges. It is quite possible that some endpoint maps to a branch point under iteration of p_c.

The map $p_c \colon T \to T$ is expanding in the following sense: for any arc $[z, z'] \subset T$ with $z \neq z'$, there is a $k \in \mathbb{N}$ such that $p_{c_0}^{\circ k}([z, z']) \ni 0$.

Indeed, the set of points in J which are in the same connected component of $J \setminus \{c_0\}$ for k iterations has diameter tending to 0 as $k \to \infty$; this follows from an easy hyperbolic contraction argument.

Every Hubbard tree T has exactly one interior fixed point, usually called α: this is a fixed point such that $T \setminus \{\alpha\}$ is disconnected. To see this, observe that every polynomial p_c of degree d has exactly $d-1$ dynamic rays which are fixed under p_c: these are the rays at angles $k/(d-1)$ for $k = 0, 1, \ldots, d-2$, and they must land at fixed points of p_c. Indeed, all their landing points must be different (otherwise, two such rays landing at a common point cut $\mathbb{C}$ into two open parts, and the part not containing the critical point would have to map to itself homeomorphically). Now p_c has exactly d fixed points in $\mathbb{C}$, counting multiplicities. If the critical orbit is preperiodic, then all d fixed points are distinct and repelling, so there remains one fixed point, called α, which must be the landing point of several periodic rays of period at least 2. Since these rays are not fixed, they must be permuted transitively by p_c [**M3**, Lemma 2.7]. This implies that $\alpha \in [0, c] \subset T$: we have $p_c([\alpha, 0]) = [\alpha, c]$, and since the rays at α are permuted transitively, it follows that $[\alpha, 0] \cap [\alpha, c] = \{\alpha\}$.

The two rays at α enclosing the critical value are the characteristic rays at α, and in parameter space they bound the wake of a hyperbolic component which bifurcates from the unique period 1 component: this wake is a subwake of the period 1 hyperbolic component, and this subwake contains the parameter c.

• Branch Points in Multibrot Sets

Of fundamental importance for the investigation of the Multibrot sets is the following Branch Theorem, which was first proved by Douady and Hubbard for the Mandelbrot set. This theorem is one of the principal results of their theory of "nervures" (Exposés XX–XXII in [**DH1**]; the Branch theorem is their Proposition XXII.3). Another proof can be found in [**LS1**, Theorem 9.1].

Theorem 3.1 (Branch Theorem).
For every two postcritically finite parameters $c \neq \tilde{c}$, exactly one of the following holds:

- *c is in the wake of $\tilde{c}$, or vice versa;*
- *there is a Misiurewicz point such that c and $\tilde{c}$ are in two different of its subwakes;*
- *there is a hyperbolic component such that c and $\tilde{c}$ are in two different of its subwakes.*

PROOF. Let ϑ be the external angle of one of the parameter rays landing at the parameter $\tilde{c}$ (if that is a Misiurewicz parameter) or at the root of the hyperbolic component with center $\tilde{c}$ (otherwise). We will work in the dynamical plane of c, which we tried to sketch in Figure 2.

Assumption. Suppose that c is a Misiurewicz parameter: the critical orbit $c_0 = 0, c_1 = c, c_2, c_3, \ldots$ is strictly preperiodic. In this case, the Julia set J is a dendrite, and for any two points $z, z' \in J$ there is a unique arc $[z, z']$ in J connecting z and z'. As defined before, the Hubbard tree of J is the union of arcs connecting the finite critical orbit. Let $\tilde{a} \in J$ be the (pre-)periodic point at which the dynamic ray $R(\vartheta)$ lands. Clearly $c \neq \tilde{a}$ (otherwise, $c = \tilde{c}$ contradicting the hypothesis).

If $c \in [0, \tilde{a}]$, then two dynamic rays land at c and, by Theorem 2.1, the parameter rays at the same angles land at c and separate the parameter ray at angle ϑ and the point $\tilde{c}$ from the origin, so $\tilde{c}$ is in the wake of c. Similarly, if $\tilde{a} \in [0, c]$ and $\tilde{a}$ is a characteristic (pre)periodic point, then c is in the wake of $\tilde{c}$. If none of these cases occurs, then $[0, \tilde{a}] \cap [0, c] = [0, b]$ for some point b which is a branch point in the Julia set, or $b = \tilde{a}$ and $\tilde{a}$ is not characteristic.

Now we claim that *the union of forward images of the arcs* $[0, c]$ *and* $[0, \tilde{a}]$ *has the topological type of a finite tree.* To see this, observe first that the union of forward images of $[0, c] = [c_0, c_1]$ equals the Hubbard tree T: it contains the critical orbit, it contains the arcs $[0, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$,..., so it is a finite connected tree spanned by the critical orbit, and this is the Hubbard tree. Finally, we join the forward images of $[0, \tilde{a}]$. Since $\tilde{a}$ has finite orbit and the orbit of 0 is in T, it follows indeed that the union of forward images of the arcs $[0, c]$ and $[0, \tilde{a}]$ is a finite tree T', say. In particular, it has finitely many branch points.

Since T' is forward invariant, all these branch points are periodic or preperiodic (or critical, but then they have finite forward orbits by assumption); in particular, the forward orbit of b is finite. If c and $\tilde{c}$ are in different subwakes of the main hyperbolic component of $\mathcal{M}_d$ of period 1, then the theorem is trivially true. We may thus assume that c and $\tilde{c}$ are in the same subwake of the period 1 component. The parameter rays bounding this subwake have the same angles as the two characteristic dynamic rays landing at the α-fixed point, so in the dynamical plane of c both c and $\tilde{a}$ are surrounded by these characteristic rays and $\alpha \in [0, b]$. In particular, $b \neq 0$.

The arc $[0, b]$ is naturally ordered so that 0 is the least and b is the largest element. Consider the set Z of characteristic (pre-)periodic points on $[0, b]$ and let a be the supremum of $Z \setminus \{b\}$. It is not clear yet

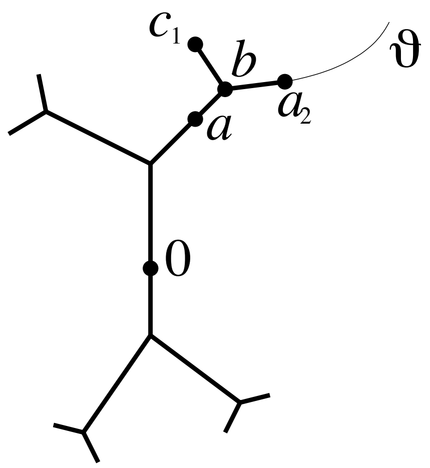

FIGURE 2. *A sketch of the dynamic plane for the parameter c as used in the proof of the Branch Theorem 3.1.*

that a is periodic or preperiodic; but if it is, then it is characteristic (as a limit of characteristic (pre-)periodic points).

(1) The case that $a \neq b$: we first show that there is a periodic point in $[a, b]$. Since the dynamics is expanding, there is a $k \geq 0$ and a $w \in [a, b]$ which maps to c after k iterations, and we may suppose that k is minimal with this property. Since a is a limit of characteristic points, the forward orbit of a does not intersect $[a, c] \setminus \{a\}$. Therefore, the k-th forward iterate of $[a, w]$ contains $[a, w]$, and by the intermediate value theorem there is a point $z \in [a, b]$ with $z \neq b$ which is fixed under the k-th iterate.

Next we show that a is periodic. This is clear if $z = a$. If $z \neq a$, then replace z by the periodic point of lowest period in the interior of $[a, b]$. By construction of a, the point z cannot be characteristic, so there is a characteristic point on the forward orbit of z. By definition, this means that there is a first number s of iterations after which z maps into $[z, c]$; clearly, s is less than the period of z. Similarly as before, the image of $[a, z]$ after s iterations contains $[a, z]$ itself, so there is a periodic point $p \in [a, z]$ with period at most s. Since the period of p is less than the period of z, we have $p \neq z$, and p cannot be in the interior of $[a, z]$ by construction of z. Therefore, $p = a$, so a is periodic with period at most s.

Since a is periodic, it is characteristic as stated above. Now a is the landing point of at least two dynamic rays, and the two characteristic rays reappear as parameter rays landing at the root of a hyperbolic component by Theorem 2.1. Looking at external angles, it follows that the points c and $\tilde{c}$ are contained in the wake of this component. If they were contained within the same subwake, the two parameter rays bounding this subwake would correspond to a characteristic periodic point in Z behind a, but by maximality of a the only possible candidate for such a point is b. The characteristic rays of b surround the critical value c_1 but not a_2 and the ray at angle ϑ, and this is a contradiction. Therefore, Theorem 2.3 implies that c and $\tilde{c}$ are contained in different subwakes.

(2) The case that $a = b$: in this case, b is characteristic. If $b = \tilde{a}$, then $\tilde{a}$ is characteristic, and this case has been treated earlier (implying that c is in the wake of $\tilde{c}$). The remaining case is that b is a branch point, hence the landing point of at least three dynamic rays which separate the points c and $\tilde{a}$ from each other and from the origin.

The point b is a limit of characteristic points in Z. If b was periodic, the dynamic rays landing at b would be permuted transitively by the first return map of b ([**M3**, Lemma 2.7], [**S1**, Lemma 2.4]), and points in Z sufficiently close to b would be mapped onto $[b, c]$, a contradiction. Therefore, b is preperiodic, and Theorem 2.1 turns the three dynamic rays landing at b into three parameter rays landing at a common Misiurewicz point such that c and $\tilde{c}$ are in two different of its subwakes because these subwakes contain the parameter rays associated to c and $\tilde{c}$.

This finishes the proof of the theorem if c is a Misiurewicz point. If c is the center of a hyperbolic component, find a Misiurewicz point c' in the wake of c and apply the previous proof to c' and $\tilde{c}$. If $\tilde{c}$ is not in the wake of c, then the branch point for c' and $\tilde{c}$ also is the branch point for c and $\tilde{c}$. □

Remark. In [**Do, S4, R**], it is shown that the Mandelbrot set $\mathcal{M}_2$ is "almost" arcwise connected: at least every hyperbolic component and every Misiurewicz point can be connected to 0 by an arc in $\mathcal{M}_2$, i.e. by an injective curve $\gamma\colon [0,1] \to \mathcal{M}_2$. The Branch Theorem implies then that arbitrary arcs in the Mandelbrot set connecting postcritically finite parameters can branch off from each other only at Misiurewicz points or within hyperbolic components. For Multibrot sets $\mathcal{M}_d$ with $d > 2$, less is known about pathwise connectivity (see [**R**]); the Branch Theorem is thus better interpreted in a combinatorial way: for two postcritically finite parameters $c_1, c_2 \in \mathcal{M}_d$ with disjoint wakes, there is a Misiurewicz point or hyperbolic component c such that c_1 and c_2 are in different subwakes of c — branch points in $\mathcal{M}_d$ are always postcritically finite.

As a first corollary, we can describe how many rational parameter rays may land at the boundary of any interior component of a Multibrot set. All the known interior components of Multibrot sets are hyperbolic components in which the dynamics has an attracting periodic orbit. It is conjectured that non-hyperbolic ("queer") components do not exist.

Corollary 3.2 (Interior Components of the Multibrot Sets). *Every hyperbolic component of a Mandelbrot or Multibrot set has infinitely many boundary points which are landing points of rational parameter rays, and all of them are accessible from inside the component. Every non-hyperbolic component has at most one boundary point which is the landing point of a parameter ray at a rational angle.*

Remark. In Section 8, we will strengthen the second statement by showing that no rational parameter ray can ever land on the boundary of a non-hyperbolic component (provided such a thing exists at all).

Proof. The statement about hyperbolic components is well known at least in the quadratic case; see [**DH1**], [**M3**, Theorem 6.5], or [**S1**, Section 5]. For the general case, see [**Eb, ES**].

Assume that two rational parameter rays $R_{\mathcal{M}}(\vartheta_1)$ and $R_{\mathcal{M}}(\vartheta_2)$ land at different points $c_1, c_2 \in \partial\mathcal{M}_d$ on the boundary of the same non-hyperbolic component. The points c_1 and c_2 cannot also be on the boundary of the same hyperbolic component: along the boundary of each hyperbolic component, there is a dense set of parabolic parameters, and these are landing points of parameter rays at periodic angles so that c_1 and c_2 cannot be connected by the same non-hyperbolic component.

If one of the ϑ_i is periodic, then the landing point of $R(\vartheta_1)$ is the root or co-root of a hyperbolic component. When applying the Branch Theorem 3.1 below to c_i, we will mean the center of this component. The corresponding centers are different even if both angles are periodic.

Since c_1 and c_2 are on the closure of the same non-hyperbolic component, there cannot be a Misiurewicz point or a hyperbolic component separating c_1 and c_2 from each other and from the origin. Therefore, by the Branch Theorem 3.1, one of these two points must be within the wake associated to the other point. Without loss of generality, assume that c_2 is within the wake of c_1. Take a third preperiodic angle ϑ_3 within the wake of c_1 but outside of the wake of c_2, and so that its landing point c_3 is a Misiurewicz point different from c_1 and c_2. If c_2 is within the wake of the Misiurewicz point c_3, then two rays landing at c_3 separate c_1 and c_2, so c_1 and c_2 cannot be on the boundary of the same non-hyperbolic component. Otherwise the Branch Theorem supplies another Misiurewicz point or hyperbolic component separating c_2 and c_3 from each other and from the origin. Call this new branch point

c_4. If $c_4 \neq c_1$, then c_4 separates c_2 from c_1, and again these two points cannot be on the boundary of a common non-hyperbolic component.

Finally, we have to deal with the case $c_4 = c_1$. If c_1 is a Misiurewicz point, then choosing ϑ_3 within the same subwake of c_1 as c_2 assures that $c_4 \neq c_1$. Otherwise, c_1 is a root or co-root of a hyperbolic component W and c_2 is in the wake of W, but $c_2 \notin \overline{W}$. By Theorem 2.3, c_2 is in some subwake of W, so the rays bounding this subwake separate c_1 and c_2. □

REMARK. The corollary can also be shown (perhaps more conceptually) using internal addresses as introduced in [**LS1**].

The most important application of the Branch Theorem is to relate local connectivity and density of hyperbolicity for Mandelbrot and Multibrot sets. We will do that in Corollary 4.6, showing also that local connectivity and triviality of all fibers is equivalent.

4. Fibers of Multibrot Sets

In this section, we introduce the fundamental concept of this paper, *fibers* of Multibrot sets, and establish their most fundamental properties. We will use parameter rays at rational angles; recall that these are known to land (at parabolic respectively Misiurewicz parameters).

DEFINITION 4.1 (Separation Line).
A separation line *is either a pair of parameter rays at rational angles landing at a common point (a ray pair), or a pair of parameter rays at rational angles landing at two different points on the boundary of the same hyperbolic component of the Multibrot set, together with a simple curve within this hyperbolic component which connects the landing points of the two rays. Two points c, c' in a Multibrot set* can be separated *if there is a separation line γ avoiding c and c' such that c and c' are in different connected components of $\mathbb{C} \setminus \gamma$.*

DEFINITION 4.2 (Fibers and Triviality).
The fiber of a point c in a Multibrot set $\mathcal{M}_d$ is the set of all points $c' \in \mathcal{M}_d$ which cannot be separated from c. The fiber of c is trivial *if it consists of the point c alone.*

It will turn out that points on the closure of a hyperbolic component of a Multibrot set have trivial fibers. For all other points, it will be good enough to construct fibers using ray pairs at periodic external angles. We will justify this in Section 8.

The idea of fibers is related to the Branner-Hubbard-Yoccoz puzzle: a typical proof of local connectivity at a point z consists in establishing *shrinking of puzzle pieces* around z. The fiber of a point contains exactly those points which are always in the same puzzle piece, independently of the puzzle chosen. Our arguments will thus never use specific puzzles. The definition here is somewhat simpler than in [**S2**], which was written for arbitrary compact connected and full subsets of $\mathbb{C}$: some conceivable difficulties cannot occur for Multibrot sets because of the Branch Theorem 3.1 and its Corollary 3.2. The equivalence of both definitions follows from Corollary 3.2 and [**S2**, Lemma 2.4].

PROPOSITION 4.3 (Fibers of Interior Components).
The fiber of a point within a hyperbolic component of a Multibrot set is always trivial. The fiber of a point within any non-hyperbolic component of a Multibrot set always contains the closure of its non-hyperbolic component.

PROOF. Let c be an interior point in a hyperbolic component. It can easily be separated from any other point within the same hyperbolic component by a separation line consisting of two rational parameter rays landing on the boundary of this hyperbolic components, and a curve within this component; there are infinitely many such rays by Corollary 3.2.

Any boundary point of the hyperbolic component, and any point in $\mathcal{M}_d$ outside of this component, can also be separated from c by such a separation line (here it is important to have more than two boundary points of the component which are landing points of rational parameter rays; if there were only two such boundary points, then these points could not be separated from any interior point).

By definition, no separation line can run through a non-hyperbolic component, so the fibers of all non-hyperbolic interior points (if any) contain at least the closure of this non-hyperbolic component. (Since no non-hyperbolic component can have more than one boundary point which is the landing point of rational rays, no separation line could run through such a component even if we had not excluded this in the definition; so the definition as written does not impose additional restrictions on possible separation lines). □

Here are a few more useful properties of fibers.

LEMMA 4.4 (Properties of Fibers).
Fibers have the following properties:

(1) *Fibers are always compact, connected and full.*
(2) *The relation "is in the fiber of" is symmetric: for two points $c, c' \in \mathcal{M}_d$, either each of them is in the fiber of the other one, or both points can be separated.*

(3) *The boundary of any fiber is contained in the boundary of* $\mathcal{M}_d$*, unless the fiber consists of a single hyperbolic parameter and is thus trivial.*

(4) *If* γ *is a separation line of* $\mathcal{M}_d$ *and* $c \in \mathcal{M}_d \setminus \gamma$*, then*

$$\{c' \in \mathcal{M}_d \colon\ \gamma \text{ does not separate } c' \text{ from } c\ \}$$

is connected.

Proof. If c' is not in the fiber of c, then by definition these two points can be separated, and c is not in the fiber of c'. This proves the second claim.

No interior point of $\mathcal{M}_d$ can be in the boundary of any non-trivial fiber: a hyperbolic interior point is a trivial fiber by itself, and closures of non-hyperbolic components are completely contained in fibers. This settles the third claim.

For the last claim, it is clear that $\mathbb{C} \setminus \gamma$ consists of two connected components, say U_1 and U_2 with $c \in U_1$. Then $M' := (U_1 \cup \gamma) \cap \mathcal{M}_d$ is closed, and we need to show that M' is connected. If not, then $M' = P \cup Q$ with two non-empty and disjoint sets P and Q which are closed in M'. But since $M' \cap \gamma$ is closed, it follows that $P \cap \gamma$ and $Q \cap \gamma$ are disjoint closed subsets of γ, and $(P \cap \gamma) \cup (Q \cap \gamma) = M' \cap \gamma$. But this is a contradiction because $\gamma \cap M' = \gamma \cap \mathcal{M}_d$ is connected.

The fiber of any $c \in \mathcal{M}_d$ is compact because every separation line separates an open subset of $\mathcal{M}_d$ from c. In the construction of the fiber of c, the set of points in $\mathcal{M}_d$ not separated from c by one ray pair is a connected and full neighborhood of c, and nested intersections of compact, connected and full subsets of $\mathbb{C}$ are compact, connected and full. The only further possible separation lines run through hyperbolic components, and it is easy to modify a sequence of such separation lines so as to leave nested intersections of compact connected and full neighborhoods of c. (See also [**S2**, Lemma 2.4].) □

A key observation is that triviality of a fiber of some $c \in \mathcal{M}_d$ implies that $\mathcal{M}_d$ is locally connected at c. Note that the converse is false: if a compact connected and full set $K \subset \mathbb{C}$ is locally connected at some $z \in \partial K$, this does not imply that the fiber of z is trivial, or that z is the landing point of any external ray (although sometimes this assumption seems to be made implicitly: triviality of the fiber of z is a strictly stronger property, and it is this property which implies the landing of some external ray). A counterexample is provided by [**M1**, Figure 36] or [**S2**, Figure 2].

Proposition 4.5 (Trivial Fibers Yield Local Connectivity). *If the fiber of a point $c \in \mathcal{M}_d$ is trivial, then $\mathcal{M}_d$ is locally connected at c. Moreover, if $c \in \partial\mathcal{M}_d$ has trivial fiber and is in the impression of the parameter ray at some angle ϑ (in particular, if the ϑ-ray lands at c), then for any sequence of external angles converging to ϑ, the corresponding impressions converge to $\{c\}$ in the Hausdorff metric, and the parameter ray at angle ϑ has impression $\{c\}$, so in particular it lands at c. If all the fibers of $\mathcal{M}_d$ are trivial, then $\mathcal{M}_d$ is locally connected, all parameter rays land, all impressions are points, and the landing points depend continuously on the angle.*

Proof. Consider a point $c \in \mathcal{M}_d$ with trivial fiber. If c is in the interior of $\mathcal{M}_d$, then $\mathcal{M}_d$ is clearly locally connected at c. Otherwise, let U be an open neighborhood of c. By assumption, every point $c' \in \mathcal{M}_d \setminus U$ can be separated from c such that the separation avoids c and c'. The region cut off from c is open; what is left is a closed neighborhood of c having connected intersection with $\mathcal{M}_d$ (Lemma 4.4 item 4). By compactness of $\mathcal{M}_d \setminus U$, a finite number of such cuts suffices to remove every point outside U, leaving another closed neighborhood of c intersecting $\mathcal{M}_d$ in a connected set. Similarly, if c is in the impression of the ϑ-ray, then parameter rays with angles sufficiently close to ϑ will have their entire impressions in $\overline{U}$ (although the rays need not land). The impression of the ϑ-ray is thus contained in the fiber of c, so the ray lands at c. Finally, it is easy to see that the impression of any ray is contained in a single fiber: the impression of the ray at angle ϑ cannot cross a separation line which does not involve the ray at angle ϑ (compare [**S2**, Lemma 2.5]). $\square$

Now we show that local connectivity of the *entire* Multibrot set is equivalent to *all* its fibers being trivial (while local connectivity at a single point *does not* imply triviality of the fiber of this point). Local connectivity resp. triviality of fibers of all of $\mathcal{M}_d$ implies that every interior component is hyperbolic. We use the ideas of the original proof of Douady and Hubbard [**DH1**, Exposé XXII.4].

Corollary 4.6 (Local Connectivity, Fibers & Hyperbolicity). *For the Mandelbrot and Multibrot sets, local connectivity at all points is equivalent to triviality of all fibers, and both imply density of hyperbolicity.*

Proof. We know from Proposition 4.5 that triviality of all fibers implies local connectivity. Moreover, any non-hyperbolic component would be contained in a single fiber, so triviality of all fibers implies that all interior components are hyperbolic. Since the exterior of $\mathcal{M}_d$ is hyperbolic as well, triviality of all fibers of $\mathcal{M}_d$ implies that hyperbolic dynamics is dense in the space of unicritical polynomials of degree d.

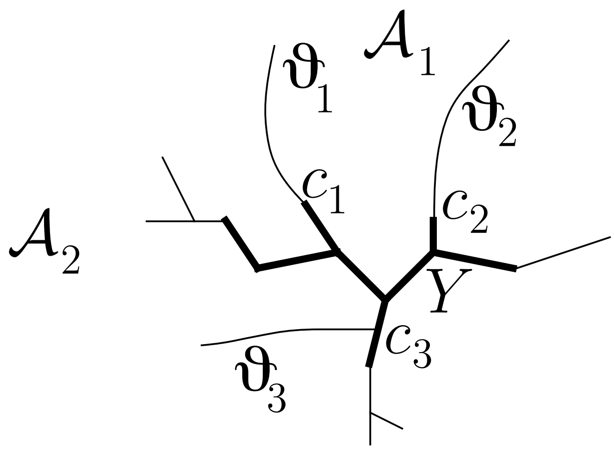

FIGURE 3. *Illustration of the proof that local connectivity implies that all fibers are trivial. Highlighted is a non-trivial fiber Y, and three of its boundary points c_1, c_2, c_3 together with parameter rays landing there. The Branch Theorem, applied to the hyperbolic components $\mathcal{A}_1$ and $\mathcal{A}_2$, provides then a contradiction.*

It remains to show that local connectivity of a Multibrot set implies that all fibers are trivial. For this, we assume that the Multibrot set $\mathcal{M}_d$ is locally connected. If there is a fiber which is not a singleton, or which even contains a non-hyperbolic component, denote it Y. The set Y is then uncountable and its boundary is contained in the boundary of $\mathcal{M}_d$ by Lemma 4.4. Let c_1, c_2, c_3 be three boundary points which are not landing points of any of the countably many parameter rays at rational angles; compare Figure 3. By local connectivity and Carathéodory's Theorem, there are three parameter rays at angles $\vartheta_1, \vartheta_2, \vartheta_3$ landing at these points, and these three rays separate $\mathbb{C} \setminus Y$ into three regions. Similarly, the three angles cut $\mathbb{S}^1$ into three open intervals. Pick one periodic angle from each of the two intervals which do not contain 0; the corresponding parameter rays land at roots or co-roots of two hyperbolic components $\mathcal{A}_1$ and $\mathcal{A}_2$. The three parameter rays at angles ϑ_i, together with Y, separate these two components from each other and from the parameter ray at angle 0 and thus from the origin. Applying the Branch Theorem 3.1 to these two components, there must either be a Misiurewicz point or a hyperbolic component separating these two components from each other and from the origin. If it is a Misiurewicz point, then three rational rays landing at it must separate the three points c_1, c_2, c_3 from each other, which is incompatible with Y being a single fiber or a non-hyperbolic component. Similarly, if the separation is given by a hyperbolic component, then this component, together with the parameter rays forming its wake and the subwakes containing $\mathcal{A}_1$ and $\mathcal{A}_2$, again separate the three c_i, yielding the same contradiction.

We conclude that, if a Multibrot set is locally connected, then its fibers are trivial, and every connected component of its interior is hyperbolic. □

Remark. Local connectivity of the entire set $\mathcal{M}_d$ is definitely stronger than density of hyperbolicity: the former amounts to fibers being points, while the latter means only that fibers have no interior. This argument is taken from Douady [**Do**].

5. Boundaries of Hyperbolic Components

In this section, we will study boundary points of hyperbolic components, except roots of primitive components which will require special treatment (see Section 6). All results in this sections are corollaries to Theorem 2.3.

Corollary 5.1 (Trivial Fibers at Hyperbolic Components).
In every Multibrot set, the fiber of every boundary point of every hyperbolic component is trivial, except possibly at the root of a primitive component. The fiber of the root contains no point within the limb of the component.

Proof. Any boundary point c of a hyperbolic component can obviously be separated from any point within the closure of the component and from every sublimb of which it is not a root. This shows that fibers are trivial for boundary points at irrational internal angles and at co-roots (which exist only for $d \geq 3$), because by Theorem 2.3 neither have sublimbs attached. If c is the root of a hyperbolic component W, then it can obviously be separated from any other point within the closure of W or from any rational sublimb, so its fiber contains no point within the limb of the component. If c is the point where the component W bifurcates from W_0, then both arguments combine to show that the fiber of c is trivial. □

The following corollary has been suggested by John Milnor. It will be strengthened in Corollary 6.4.

Corollary 5.2 (Roots Do Not Disconnect Limbs).
The limb of any hyperbolic component is connected. □

Remark. Note that we define the wake of a hyperbolic component to be open, so that the limb of the component (the intersection of the wake with the Multibrot set) is a relatively open subset of $\mathcal{M}_d$. Sometimes, wake and limb are defined so as to also contain the root. With that definition, the corollary says that the root does not disconnect the limb.

Proof. Let W be the hyperbolic component defining the wake. Assume that its limb consists of more than one connected component, and let K be a connected component not containing W. Any point within K must then, by Theorem 2.3, be contained within some rational sublimb of W, and all of K must be contained within the same sublimb. But since the entire set $\mathcal{M}_d$ is connected and the limb is obtained from $\mathcal{M}_d$ by cutting along a pair of parameter rays landing at the root of the limb, it follows that the limb is connected as well. □

Here is another corollary which has been found independently by Lavaurs [**La**, Proposition 1], Hubbard (unpublished) and Levin [**Le**, Theorem 7.3]; another proof is in [**LS1**, Lemma 3].

Corollary 5.3 (Analytic Continuation Over the Entire Wake).
If $c \in \mathcal{M}_d$ is a parameter which has an attracting periodic point z, then this periodic point can be continued analytically as an analytic map $z(c)$ over the entire wake of the hyperbolic component containing c, and it is repelling except on the closure of the component. Within any subwake at internal angle p/q, the point $z(c)$ is the landing point of exactly q dynamic rays with combinatorial rotation number p/q.

Proof. Denote the hyperbolic component by W, let c_0 be its root and let U be the wake of W. The orbit $z(c)$ can be continued analytically throughout a neighborhood of $\overline{W} \setminus \{c_0\}$ restricted to the wake of W, and the orbit becomes indifferent on ∂W ([**M3**, Section 6], [**S1**, Section 5]).

Let $U_{p/q}$ be a subwake of W at rational internal angle p/q, and let $c_{p/q} \in \partial W$ be the root point of $U_{p/q}$. Then $z(c)$ can be continued analytically in a neighborhood of $c_{p/q}$, and within $U_{p/q}$ it is the repelling landing point of q periodic dynamic rays. The locus where these q rays land together at a repelling orbit is exactly $U_{p/q}$ [**M3**], so $z(c)$ can be continued analytically throughout $U_{p/q}$. By Theorem 2.3, the limb of W consists of $\overline{W} \setminus \{c_0\}$, together with the sublimbs at rational internal angles, so $z(c)$ is repelling within all of $U \setminus \overline{W}$. While this argument a priori holds only for analytic continuation along curves within $U_{p/q}$, the wake U is simply connected, so the homotopy class of curves of analytic continuation within U does not matter. (However, analytic continuation within $\mathbb{C}$ can bring the orbit $z(c)$ to any other orbit of period n [**LS1**].) $\square$

Remark. For parameters within a hyperbolic component of period n, there are d distinguished periodic points: the point on the attracting orbit in the Fatou component containing the critical value, and $d-1$ further boundary points of the same Fatou component with the same ray period (and possibly smaller orbit period). If $n > 1$, then exactly one of these boundary points is the landing point of at least two dynamic rays (this point is the "dynamic root" of the Fatou component), it is repelling and can be continued analytically over the entire wake of the component, retaining its dynamic rays [**M3**, Theorem 3.1]. The remaining $d-2$ distinguished boundary points of the Fatou component are landing points of one dynamic ray each (they are "dynamic co-roots"), and they also remain repelling and keep their rays throughout the wake [**Eb, ES**]. For the attracting orbit, this is not true: the orbit remains repelling away from $\overline{W}$, but the rays landing at the orbit change: traversing the wake of W outside of $\mathcal{M}_d$, the combinatorial rotation number of the orbit $z(c)$ behaves (locally) monotonically with respect to external angles, rotating $d-1$ times around $\mathbb{S}^1$. See also the appendices in [**M3**] and [**GM**].

6. Roots of Hyperbolic Components

In this section, we prove triviality of fibers at roots of primitive components. This case is not handled by the arguments from the previous section, nor by the Yoccoz inequality. We start with a combinatorial preparation.

Lemma 6.1 (Approximation of Ray Pairs, Periodic Case). *Let $\vartheta < \vartheta'$ be the two periodic external angles of the root of a primitive hyperbolic component of period $n > 1$. Then there exists a sequence of periodic parameter ray pairs $(\vartheta_k, \vartheta'_k)$ such that $\vartheta_k \nearrow \vartheta$ and $\vartheta'_k \searrow \vartheta'$.*

Proof. Let c be the center of the hyperbolic component. In the dynamical plane of p_c, the two dynamic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together at the Fatou component containing the critical value; let U_1 be the open region thus separated from the origin. Since (ϑ, ϑ') is a characteristic ray pair, U_1 does not contain a ray on the forward orbit of ϑ or ϑ', hence U_1 does not contain a point on the critical orbit other than the critical value c.

The preimage of the ray pair (ϑ, ϑ') consists of d ray pairs which bound a neighborhood U_0 of the critical point. Then $p_c\colon U_0 \to U_1$ is a branched cover of degree d, ramified only over the critical value c. Pulling back along the periodic critical orbit, we obtain a neighborhood U_{-n+1} of c such that $p_c^{\circ(n-1)}\colon U_{-n+1} \to U_0$ is a conformal isomorphism, and U_{-n+1} is bounded by d ray pairs, each of which maps to (ϑ, ϑ') under $p_c^{\circ n}$.

Let $V := \mathbb{C} \setminus (U_1 \cup U_0 \cup \ldots \cup U_{-n+1})$, and let V_0 be the connected component of V containing the ray pair (ϑ, ϑ') on its boundary. Since the component is primitive, the orbits of ϑ and ϑ' are disjoint, so V_0 is non-empty. By construction, V does not contain any point on the critical orbit, so V_0 can be pulled back arbitrarily often along any branch of p_c^{-1}. Since the periodic ray pair (ϑ, ϑ') is on the boundary of V_0, there is a branch of $p_c^{\circ(-n)}$ sending V_0 onto some $V_1 \subset \mathbb{C}$ such that V_1 contains (ϑ, ϑ') on its boundary.

Now we show that $V_1 \subset V_0$. Clearly $V_1 \cap V_0$ is non-empty because both ϑ and ϑ' have period n, and if $V_1 \not\subset V_0$, then V_1 would contain a dynamic ray bounding V_0. Mapping forward n times, V_0 would contain a dynamic ray on the forward orbit of ϑ or ϑ', but all those are on the boundary of some U_i.

Since $V_1 \subset V_0$, the map $p_c^{\circ(-n)}\colon V_0 \to V_1$ is a uniform contraction with respect to the hyperbolic metric on $\mathbb{C} \setminus \{0, c, p_c(c), \ldots\}$, and it fixes the landing point z_1 of the ray pair (ϑ, ϑ'), so iteration of this map converges locally uniformly on V_0 to z_1. Let $V_1, V_2, V_3 \ldots$ denote the respective image domains of V_0. All these domains are bounded by the same number of ray pairs: one of them is (ϑ, ϑ'), all the others are preperiodic on the backwards orbit of (ϑ, ϑ'). Therefore, each V_k contains one preperiodic boundary ray pair $(\vartheta_k, \vartheta'_k)$ which separates V_k from U_0 and hence from the origin, and possibly after relabeling we may assume that $\vartheta_k \nearrow \vartheta$ and $\vartheta'_k \searrow \vartheta'$.

For every k, there is an $s < n$ such that $p_c^{\circ kn+s}$ sends $(\vartheta_k, \vartheta'_k)$ to (ϑ, ϑ').[1] More precisely, it is easily seen that $p_c^{\circ kn+s}(\vartheta_k) = \vartheta'$ and $p_c^{\circ kn+s}(\vartheta'_k) = \vartheta$: this is because the external angles of rays within V_0 and V_k are contained within finitely many open intervals, and ϑ' and ϑ_k are lower boundaries of such intervals, while ϑ_k and ϑ'_k are upper boundaries. Therefore, we can pull back V_0 for $kn + s$ steps along the backwards orbit from (ϑ, ϑ') to $(\vartheta_k, \vartheta'_k)$ so as to yield a domain $\tilde{V}_k \subset V_k$ with a conformal isomorphism $p_c^{\circ(kn+s)}\colon \tilde{V}_k \to V_0$.

The set of external angles of each V_k and each $\tilde{V}_k$ consists of the same number of closed intervals. Let I_k be the boundary interval of $\tilde{V}_k$ with ϑ_k as lower boundary, and let I'_k be the boundary interval of $\tilde{V}_k$ with ϑ'_k

[1] The value of s does not depend on k at least for large k, but we do not need this fact.

as upper boundary. Since the dynamics of p_c induces the doubling map on external angles, the set $2^{kn+s}I_k$ is an interval of boundary angles of V_0 with ϑ' as lower boundary; therefore, $2^{kn+s}I_k \supset I'_k$. Similarly, $2^{kn+s}I'_k \supset I_k$. Therefore, there are external angles $\tilde\vartheta_k \in I_k$ and $\tilde\vartheta'_k \in I'_k$ with $2^{nk+s}\tilde\vartheta_k = \tilde\vartheta'_k$ and $2^{nk+s}\tilde\vartheta'_k = \tilde\vartheta_k$, so both angles are periodic with period $2(nk+s)$ and the corresponding rays $R(\tilde\vartheta_k)$ and $R(\tilde\vartheta'_k)$ land at repelling periodic points z_{nk+s} and z'_{nk+s}. Since $p_c^{\circ(-2(nk+s))}$ induces a conformal map from V_0 into $V_k \subset V_0$ which fixes the two rays $\tilde\vartheta_k$ and $\tilde\vartheta'_k$, the same branch fixes z_{nk+s} and z'_{nk+s}; but that implies $z_{nk+s} = z'_{nk+s}$, so $(\tilde\vartheta_k, \tilde\vartheta'_k)$ is a ray pair consisting of periodic angles. If it is not characteristic, then the characteristic ray pair of the orbit separates $(\tilde\vartheta_k, \tilde\vartheta'_k)$ from c, and thus also from (ϑ, ϑ'). Possibly after relabeling, we may assume that the ray pairs $(\tilde\vartheta_k, \tilde\vartheta'_k)$ are characteristic, so by Theorem 2.1 there is a corresponding periodic ray pair in parameter space which satisfies the claim. □

Remark. The previous lemma can also be proved using Hubbard trees, similarly as in Lemma 7.1 (see [**S3**]). Our proof is motivated by the Orbit Separation Lemma from [**S1**, Lemma 3.7] and [**Eb**].

Lemma 6.2 (Fiber of Primitive Root).
Let c_0 be the root of any primitive hyperbolic component and let $\tilde c \neq c_0$ be some point in $\mathcal{M}_d$ which is not in the limb of c_0. Then there is a parameter ray pair at rational angles separating c_0 from $\tilde c$ and from the origin. In particular, the fiber of the root of a primitive hyperbolic component does not contain any point outside of the wake of the component.

Proof. The strategy of the proof is simple: if we denote the periodic parameter ray pair landing at c_0 by (ϑ, ϑ'), then Lemma 6.1 supplies a sequence of parameter ray pairs $(\vartheta_k, \vartheta'_k)$ converging to (ϑ, ϑ'), and in the dynamics of c_0 there are characteristic dynamic ray pairs at the same angles. If not all of these characteristic dynamic ray pairs exist for $\tilde c$, then $\tilde c$ is separated from c_0 by one of the parameter ray pairs $(\vartheta_k, \vartheta'_k)$ (because these ray pairs bound the regions in parameter space for which the corresponding dynamic ray pairs exist: see Theorem 2.1). In this case, the two points c_0 and $\tilde c$ can be separated as claimed and are thus in different fibers.

We may hence assume that all these dynamic ray pairs exist for the parameter $\tilde c$. We will then show that the limiting dynamic rays at angles ϑ and ϑ' also form a ray pair: this forces $\tilde c$ to be in the closure of the wake of c_0, contradicting our assumption. This finishes the proof of the lemma.

The only thing we need to prove, then, is the following claim: *assume that for the parameter $\tilde{c}$, the dynamic rays $R(\vartheta_k)$ and $R(\vartheta'_k)$ land together for every k. Then the dynamic rays $R(\vartheta)$ and $R(\vartheta')$ also land together.*

To prove this claim, denote the period of ϑ and ϑ' by n. We start the discussion in the dynamical plane of c_0. For $k \geq 0$, let $U_k \subset \mathbb{C}$ be the domain enclosed by the ray pairs (ϑ, ϑ') and $(\vartheta_k, \vartheta'_k)$. Clearly, $U_{k+1} \subset U_k$ for all k. Choose $N \in \mathbb{N}$ large enough so that U_N does not contain a point on the critical orbit. Set $p_0(z) := z^d + c_0$. Then there is a single branch of p_0^{-n} which fixes the dynamic rays $R(\vartheta)$ and $R(\vartheta')$ and which maps each U_k into itself (at least for $k \geq N$).

All this works in the dynamical plane of c_0. For the parameter $\tilde{c}$, denote the landing points of $R(\vartheta)$ and $R(\vartheta')$ by w and w', respectively. Then the critical value cannot be separated from w or w' by any rational ray pair (α, α') landing at a repelling orbit: there is no such ray pair for c_0; if there exists one for $\tilde{c}$, then such a ray pair would continue to exist in a neighborhood of $\tilde{c}$, and the largest neighborhood of $\tilde{c}$ where such a ray pair exists will be bounded by parameter ray pairs at rational angles [**M3**, Theorem 3.1].

Similarly, for any $m > 0$, the m-th forward image of the critical value cannot be separated from the landing points of the dynamic rays $R(2^m\vartheta)$ and $R(2^m\vartheta')$, or we could pull back such a separating ray pair, obtaining a separation between the critical value and $R(\vartheta)$ and $R(\vartheta')$. It follows that each $(\vartheta_k, \vartheta'_k)$ (for $k > N$) is separated from the critical orbit by other such ray pairs $(\vartheta_j, \vartheta'_j)$ and $(\vartheta_{j'}, \vartheta'_{j'})$ with $j < k < j'$.

Set $p(z) := z^d + \tilde{c}$, $P := \overline{\cup_{m \geq 0} p^{\circ m}(c)}$ (the postcritical set), $X := \mathbb{C} \setminus P$ and $X' := p^{-1}(X)$. The set X is clearly open, and it is non-empty because it contains the region bounded by the ray pairs $(\vartheta_N, \vartheta'_N)$ and $(\vartheta_k, \vartheta'_k)$ for $k > N$. Since P is forward invariant, we have $X' \subset X$, and $p\colon X' \to X$ is an unbranched covering, i.e., a local isometry with respect to the hyperbolic metrics of X and X'.

Let u, u' be two points on the dynamic rays $R(\vartheta)$ and $R(\vartheta')$, respectively, and construct a simple piecewise analytic curve γ_N as follows: connect u along an equipotential to $R(\vartheta_N)$ (the "short way", decreasing angles), then continue along this ray towards its landing point, then out along $R(\vartheta'_N)$ up to the potential of u', and connect finally along this equipotential to u' (increasing angles). This curve runs entirely within X, so it has finite hyperbolic length ℓ_N, say.

Now we pull this curve back along the periodic backwards orbit of $R(\vartheta)$ and $R(\vartheta')$, yielding a sequence of curves $\gamma_N, \gamma_{N+1}, \gamma_{N+2}, \dots$ after $0, k, 2k, \dots$ steps. The branch of the pull-back fixing ϑ will also fix ϑ': we had verified that in the dynamics of c_0, and this is no different for $\tilde{c}$ because the critical value is always on the same side of all the ray pairs.

Denote the hyperbolic lengths of γ_k in X by ℓ_k. Since $X' \subset X$, this sequence is strictly monotonically decreasing. The two endpoints of the curves in the sequence obviously converge to the two landing points w and w' of $R(\vartheta)$ and $R(\vartheta')$. Now there are two possibilities: either the sequence (γ_k) has a subsequence which stays entirely within a compact subset of X, or it does not. If it does, then the hyperbolic length shrinks by a definite factor each time the curve is within the compact subset of X, so that we can connect points arbitrarily closely to w and w' (with respect to the Euclidean metric) by (hyperbolically) arbitrarily short curves at bounded (Euclidean) distances from the postcritical set, and this implies $w = w'$ as required. However, if these curves converge to the boundary, then their Euclidean lengths must shrink to zero, and we obtain the same conclusion. The proof of the lemma is complete. □

Corollary 6.3 (Trivial Fibers at Hyperbolic Components). *The fiber of every point on the closure of any hyperbolic component of a Multibrot set $\mathcal{M}_d$ is trivial.*

Proof. In Corollary 5.1, we have already done most of the work; the remaining statement is exactly the content of the previous lemma. □

The next corollary strengthens Corollary 5.2 and has also been suggested by Milnor.

Corollary 6.4 (Roots Disconnect Multibrot Sets). *Every root of a hyperbolic component of period greater than* 1 *disconnects its Multibrot set into exactly two connected components.*

Proof. Let c_0 be the root of a hyperbolic component W of period greater than 1. It is the landing point of exactly two parameter rays at periodic angles. This parameter ray pair disconnects $\mathbb{C}$ into exactly two connected components. The component not containing the origin is the *wake* of W; its intersection with $\mathcal{M}_d$ is the limb. By Corollary 5.2, the limb is connected. Now let $\tilde{c} \in \mathcal{M}_d$ with $\tilde{c} \neq c_0$ be any parameter not in the limb of W.

We first discuss the case when W is primitive. By Lemma 6.2 above, there is a rational parameter ray pair S separating c_0 from $\tilde{c}$ and from the origin. This ray pair S disconnects $\mathbb{C}$ into two connected components. Let $\mathcal{M}'$ be the closure of the connected component of the origin intersected with $\mathcal{M}_d$. Then $\mathcal{M}'$ itself is connected and contains the origin and $\tilde{c}$. It follows that $\tilde{c}$ and 0 are in the same connected component of $\mathcal{M}_d \setminus \{c_0\}$, so c_0 cuts $\mathcal{M}_d$ into exactly two connected components.

In the non-primitive case, the parameter c_0 is the root of W and on the boundary of another hyperbolic component, say W_0. Then by Theorem 2.3, there are three cases: the point $\tilde{c}$ may be outside of the limb of W_0 so that the two parameter rays landing at the root of W_0 separate c_0 and $\tilde{c}$; the point $\tilde{c}$ may be on the closure of W_0; or it may be in a sublimb of W_0 at rational internal angles, but not in the sublimb with c_0 on its boundary (which is the wake of c_0). In all three cases, it is easy to see that $\tilde{c}$ must be in the same connected component of $\mathcal{M}_d \setminus \{c_0\}$ as the origin.

Therefore, c_0 disconnects $\mathcal{M}_d$ into exactly two connected components in the primitive as well as in the non-primitive case. □

Corollary 6.5 (Rays at Boundary of Hyperbolic Component). *Every boundary point of a hyperbolic component at irrational internal angle is the landing point of exactly one parameter ray, and the external angle of this ray is irrational (and in fact transcendental).*

Every boundary point at rational internal angle is the landing point of exactly two parameter rays, except co-roots: these are the landing points of exactly one parameter ray. The external angles of all these rays are periodic and in particular rational.

In no case is a boundary point of a hyperbolic component in the impression of any further parameter ray.

Proof. Every boundary point of a hyperbolic component at irrational internal angle is in the boundary of $\mathcal{M}_d$ and thus in the impression of some ray. Since the fiber of this boundary point is trivial, the ray must land there and its impression is a single point (Proposition 4.5). And since all the parameter rays at rational rays land elsewhere, the external angle of the ray must be irrational (and in fact transcendental; see [**BuS**] for a proof in the quadratic case). If this point is in the impression of a further ray, this ray must land there, too, and these two rays separate some open subset of $\mathbb{C}$ from the component. Since not all rays between the two landing rays can land at the same point (the rational rays land elsewhere), there must be some part of $\mathcal{M}_d$ between these two rays, and this contradicts Theorem 2.3.

We know that boundary points at rational internal angles are parabolic points, and the statements about the landing properties of rational parameter rays are well known (compare Section 2). If a parabolic point is in the impression of an irrational parameter ray, it is the landing point of that ray. We then get a similar contradiction as above, except if the parabolic point is the root of a primitive component and the extra parameter ray is outside of the wake of the component. But that case is handled conveniently by Lemma 6.1, supplying lots of parameter ray pairs which separate any parameter ray outside of the wake of the component from the component, its root, and all of its co-roots. □

7. Misiurewicz Points

Now we turn to Misiurewicz points, proving triviality of fibers in a somewhat similar way to Section 6. Again, we need some combinatorial preparations in analogy to Lemma 6.1.

Lemma 7.1 (Approximation of Ray Pairs, Preperiodic Case). *Let c_0 be a Misiurewicz parameter with external angles $\vartheta_0 < \vartheta_1 < \ldots < \vartheta_s$. Then for every sufficiently small $\varepsilon > 0$ and every $i \in \{0, 1, \ldots, s-1\}$, there is a periodic parameter ray pair (α_i, α_i') with*

$$\vartheta_i < \alpha_i < \vartheta_i + \varepsilon < \vartheta_{i+1} - \varepsilon < \alpha_i' < \vartheta_{i+1} \tag{1}$$

and similarly, there is another periodic parameter ray pair (α, α') with $\vartheta_0 - \varepsilon < \alpha < \vartheta_0$ and $\vartheta_s < \alpha' < \vartheta_s + \varepsilon$.

Remark. The number of preperiodic parameter rays landing at c_0 is $s+1$, and this can be any positive integer. If $s = 0$, then condition (1) is void.

Sketch of proof. For the postcritically preperiodic parameter c_0, we will use the Hubbard tree T as introduced in Section 2. (The argument could also be translated into statements on external angles similarly as in Lemma 6.1, thus using fewer topological properties of the Julia set.)

By expansivity of Hubbard trees, there is a sequence $(z_k) \in [0, c_0]$ of points with $z_k \to c_0$ such that $p_{c_0}^{\circ N_k - 1}(z_k) = 0$ for some sequence N_k of integers (which necessarily tends to ∞). We may assume that the z_k are close enough to c_0 so that there is no branch point of T in the interior of $[z_k, c_0]$. After replacing z_k by another point on $[z_k, c_0]$ if necessary, we may suppose $p_{c_0}^{\circ N_k} \colon [z_k, c_0] \to p_{c_0}^{\circ N_k}([z_k, c_0]) = [c_0, p_{c_0}^{\circ N_k}(c_0)]$ is a homeomorphism. Since c_0 is an endpoint of T, it follows that $p_{c_0}^{\circ N_k}$ sends $[z_k, c_0]$ over itself in an orientation reversing way, and there is a point $p_k \in [z_k, c_0]$ with $p_{c_0}^{\circ N_k}(p_k) = p_k$. The periods of p_k tend to ∞ as $k \to \infty$. The characteristic point of the orbit of p_k is either p_k itself or another point on $[p_k, c_0]$; relabel so that all p_k are characteristic. Clearly, each p_k is the landing point of a periodic ray pair at angles α_k, α'_k with $0 < \alpha_k < \vartheta_0 < \vartheta_s < \alpha'_k$; for k sufficiently large depending on ε, we even have $\vartheta_0 - \varepsilon < \alpha_k < \vartheta_0 < \vartheta_s < \alpha'_k < \vartheta_s + \varepsilon$. Now Theorem 2.1 transfers these ray pairs into parameter space. This proves the second claim of the lemma, which is all we need to do if $s = 0$.

If $s > 0$, then pick any $i \in \{0, 1, \ldots, s-1\}$ and take a Misiurewicz point c with external angles in $(\vartheta_i, \vartheta_{i+1})$. By Theorem 2.1, in the dynamics of c there is a characteristic preperiodic point z at which all the dynamic rays at angles ϑ_j land. We can find a point $w \in [z, c]$ arbitrarily close to z with the following properties:

- w is a precritical point, i.e. there is an N such that $p_c^{\circ N}(w) = c$;
- the restriction $p_c^{\circ N} \colon [z, w] \to [p_c^{\circ N}(z), p_c^{\circ N}(w)]$ is a homeomorphism (if not, then there is a $w' \in [z, w]$ which can be used instead of w);
- none of the finitely many branch points of T are in the interior of $[z, w]$ (again, otherwise w can be replaced by some $w' \in [z, w]$).

Given such a w with $p_c^{\circ N}(w) = c$, there is an $N' \le N$ such that $w \in [z, p_c^{\circ N'}(w)]$: $N' = N$ certainly works, but choose $N' > 0$ minimal. Then $p_c^{\circ N'}([z, w]) \supset [z, w]$ (since z is characteristic and preperiodic, we never have $z \in [p_c^{\circ k}(z), 0]$ for $k > 0$). Hence there is a $p \in [z, w]$ with $p_c^{\circ N'}(p) = p$. Clearly p is periodic, and minimality of N' assures that p is characteristic. By choosing w sufficiently close to z depending on $\varepsilon > 0$, we can assure that all external angles of p are in $[\vartheta_i, \vartheta_i + \varepsilon] \cup [\vartheta_{i+1} - \varepsilon, \vartheta_{i+1}]$. Again using Theorem 2.1, there is a parameter ray pair at the characteristic angles of p. □

The following obvious corollary is an analogue to Corollary 6.5.

Corollary 7.2 (No Irrational Rays at Misiurewicz Points). *No Misiurewicz point is in the impression of a parameter ray at an irrational external angle.* □

The following theorem is well known for the Mandelbrot set (i.e. $d = 2$). As far as I know, it was first proved by Yoccoz using his puzzle techniques. A variant of this proof for many cases is indicated in Hubbard's paper [**H**, Theorem 14.2]. Tan Lei has published another proof in [**TL1**].

THEOREM 7.3 (Misiurewicz Points Have Trivial Fibers).
The fiber of any Misiurewicz point $c_0 \in \mathcal{M}_d$ is trivial. In fact, every $c \in \mathcal{M}_d \setminus \{c_0\}$ can be separated from c_0 by a parameter ray pair at periodic angles.

PROOF. Let $\vartheta_0, \ldots, \vartheta_s$ be the external angles of c_0. We have to show that, for every $c \in \mathcal{M}_d \setminus \{c_0\}$, there is a rational ray pair separating c from c_0.

In the dynamic plane of c_0, let z be a periodic postcritical point, and pick N such that $p_{c_0}^{\circ N}(c_0) = z$. In a parameter neighborhood $V \ni c_0$, the point z can be continued analytically as a periodic point $z(c)$.

Let J be the Julia set for the parameter c_0. The critical value is the landing point of the $s+1$ preperiodic dynamic rays $\vartheta_0, \ldots, \vartheta_s$, hence z is the landing point of $s+1$ periodic dynamic rays and $J \setminus \{z\}$ consists of exactly $s+1$ connected components $J_0, \ldots, J_s$. In each connected component J_i, there is a periodic or preperiodic point p_i separating z from all points on the critical orbit within J_i; we may suppose that the p_i are not on the grand orbit of z: this can be proved in a similar way as in Lemma 7.1. Let J_z be the connected component of z within $J \setminus \{p_0, \ldots, p_s\}$. Since z is repelling periodic, there is an $M > 0$ such that $p_{c_0}^{\circ(-M)}(\overline{J}_z) \subset J_z$ (choosing the branch of $p_{c_0}^{\circ(-M)}$ which fixes z), and the continued pull-back of $\overline{J}_z$ along the periodic orbit of z shrinks to the point z alone: all this argument requires is to have a simply connected neighborhood of z without postcritical points. It follows that every $z' \in J \setminus \{z\}$ is separated from z by a point on the backwards orbit of one of the p_i.

Possibly by shrinking V, we may suppose that all the finitely many points on the forward orbits of the p_i retain all their dynamic rays for parameters from V, and the first M postcritical points are separated from $z(c)$ by the p_i. Then for all $c \in V$, the set $J_{z(c)}$ defined analogously as above satisfies $p_{c_0}^{\circ(-M)}(\overline{J}_{z(c)}) \subset J_{z(c)}$ because this condition is encoded in the external angles of the dynamic rays landing at the p_i. Since $z(c)$ is repelling, it follows again that every $z' \in J_c \setminus \{z(c)\}$ is separated from $z(c)$ by a point on the backwards orbit of one of the p_i (so the fiber of $z(c)$ is still trivial).

For the parameter c_0, the critical value c_0 is preperiodic, and analytic continuation gives a preperiodic point $\tilde{z}(c)$ defined on V with $\tilde{z}(c_0) = c_0$ (possibly by shrinking V again). All dynamic rays $\vartheta_0, \ldots, \vartheta_s$ land at $\tilde{z}(c)$ for $c \in V$. Now pick a $c \in V \setminus \{c_0\}$. Since separation lines cannot run through non-hyperbolic components, and every hyperbolic parameter can be separated from c_0 by Proposition 4.3 (or Lemma 7.1), we may as well suppose that $c \in \partial\mathcal{M}_d$. In the dynamics of c, the point $p_c^{\circ N}(c) \neq z(c)$ is separated from $z(c)$ by a rational ray pair, and hence c is separated from $\tilde{z}(c)$ by a rational ray pair (α, α'). Find $\varepsilon > 0$ such that $|\alpha - \vartheta_i| > 2\varepsilon$ and $|\alpha' - \vartheta_i| > 2\varepsilon$ for all i.

By Lemma 7.1, there are finitely many parameter ray pairs (α_i, α_i') separating c_0 from all parameter rays $R_{\mathcal{M}}(\vartheta)$ with $|\vartheta - \vartheta_i| > \varepsilon$ for all i. The separating dynamic ray pair (α, α') is stable under perturbations, so for parameters $c' \in \mathbb{C} \setminus \mathcal{M}_d$ sufficiently close to c the critical value has a certain external angle ϑ with $|\vartheta - \vartheta_i| > \varepsilon$ for all i. Every such perturbed parameter is thus separated from c_0 by one of the finitely many ray pairs (α_i, α_i'), and since this holds for arbitrarily small perturbations, c must be separated from c_0 as well by such a parameter ray pair. □

Corollary 7.4 (Misiurewicz Points Disconnect).
Every Misiurewicz point disconnects its Multibrot set into exactly as many connected components as there are rational parameter rays landing at it.

Proof. All the parameter rays landing at a Misiurewicz point $c_0 \in \mathcal{M}_d$ have preperiodic external angles by Corollary 7.2, and they obviously disconnect $\mathcal{M}_d$ into at least as many connected components as there are such rays.

By Lemma 7.1, between any two adjacent external rays of c_0 there is a collection of rational parameter ray pairs exhausting the interval of external angles in between, so any extra connected component at c_0 must be in the impression of the parameter rays landing at c_0, but since the fiber of c_0 is trivial, these impressions consist of the point c_0 only. □

8. Fibers and Combinatorics

Now that we have trivial fibers at all the landing points of rational rays, it follows that fibers have a number of convenient properties. These are discussed in this section, and fibers are linked to the important concept of *combinatorial classes*.

Theorem 8.1 (Fibers of $\mathcal{M}_d$ are Equivalence Classes).
The fibers of any two points in $\mathcal{M}_d$ are either equal or disjoint.

Proof. The relation "c_1 is in the fiber of c_2" is always reflexive, and it is symmetric for $\mathcal{M}_d$ by Lemma 4.4. In order to show transitivity, assume that two points c_1 and c_2 are both in the fiber of c_0. If they are not in the fibers of each other, then the two points can be separated by a separation line avoiding c_1 and c_2. If such a separation line can avoid c_0, then these two points cannot both be in the fiber of c_0. The only separation between c_1 and c_2 therefore runs through the point c_0, so c_0 cannot be in the interior of $\mathcal{M}_d$ and rational rays land at c_0. Therefore, the fiber of c_0 consists of c_0 alone. Any two points with intersecting fibers thus have indeed equal fibers. □

The theorem allows to simply speak of *fibers* of $\mathcal{M}_d$ as equivalence classes of points with coinciding fibers, as opposed to "fibers of c" for $c \in \mathcal{M}_d$. There is a map from external angles to fibers of $\mathcal{M}_d$ via impressions of external rays (compare [**S2**, Lemma 2.7]). This map is surjective onto the set of fibers meeting $\partial\mathcal{M}_d$.

The following corollary is now obvious and just stated for easier reference. It strengthens Corollary 3.2.

Corollary 8.2 (Queer Components Rationally Invisible).
No rational parameter ray lands on the boundary of a non-hyperbolic component. □

Combinatorial building blocks of the Multibrot sets which are often discussed are combinatorial classes. We will show that they are closely related to fibers.

Definition 8.3 (Combinatorial Classes and Equivalence).
We say that two connected Julia sets are combinatorially equivalent *if in both dynamic planes external rays at the same rational angles land at common points. Equivalence classes under this relation are called* combinatorial classes.

Remark. In the language of Thurston [**T**], combinatorially equivalent Julia sets are those having the same rational lamination. The definition is such that topologically conjugate (monic) Julia sets are also combinatorially equivalent (for an appropriate choice of one of the $d-1$ fixed rays to have external angle 0). In particular, all the Julia sets within any hyperbolic component, at its root and at its co-roots are combinatorially equivalent.

With the given definition, the combinatorial class of a hyperbolic component also includes its boundary points at irrational angles, although the dynamics will be drastically different there. Therefore, the definition of combinatorial equivalence is sometimes refined.

Remark. There are homeomorphic Julia sets which are not topologically conjugate and which are thus not combinatorially equivalent: as an example, it is not hard to see that the Julia set of $z^2 - 1$ (known as the "Basilica") is homeomorphic to any locally connected quadratic Julia set with a Siegel disk of period one. However, this homeomorphism is obviously not compatible with the dynamics, and the Basilica is in a different combinatorial class than any Siegel disk Julia set.

Proposition 8.4 (Combinatorial Classes and Fibers).
Hyperbolic components together with their roots, co-roots and irrational boundary points form combinatorial classes. Every other combinatorial class is equal to a single fiber. In particular, if there is any non-hyperbolic component, then its closure is contained in a single combinatorial class and a single fiber.

Proof. The landing pattern of dynamic rays changes upon entering the wake of a hyperbolic component, so any parameter ray pair at periodic angles separates combinatorial classes. Similarly, the landing pattern is different within all the subwakes of any Misiurewicz point c_0, and it is yet different in the wake exterior: there is a neighborhood V of c_0 in which the critical value of c_0 can be continued analytically as a preperiodic point $z(c)$ (which equals the critical value only for the parameter c_0). For parameters in V, the rays landing at $z(c)$ have the same angles, but the d preperiodic inverse images carry different angles according to which subwake of the Misiurewicz point the parameter is in. Hence parameter ray pairs at preperiodic angles also separate combinatorial classes, and combinatorial classes are just what fibers would be if separations were allowed only by rational ray pairs, excluding separation lines containing curves through interior components. Since fibers are constructed using a larger collection of separation lines, it follows that every fiber is contained in a combinatorial class. The closure of any non-hyperbolic component is contained in a single fiber (Corollary 3.2), so it is also contained within a single combinatorial class.

We know that the rational landing pattern is constant throughout hyperbolic components and at its root and co-roots, as well as at its irrational boundary points. Hyperbolic components with these specified boundary points are therefore in single combinatorial classes; since every further point in the Multibrot set is either in a rational subwake of the component or outside the wake of the component, the combinatorial class of any hyperbolic component is exactly as described.

Finally, we want to show that every non-hyperbolic combinatorial class is a single fiber. If this was not so, then two points within the combinatorial class could be separated by a separation line. Unless these points are both on the closure of the same hyperbolic component, such a separation line can always be chosen as a ray pair at rational angles, and we have seen above that such ray pairs separate combinatorial classes. □

The following corollary is closely related to the Branch Theorem 3.1.

Corollary 8.5 (Three Rays at One Fiber).
If three parameter rays of a Multibrot set accumulate at the same fiber, then the three rays are preperiodic and land at a common Misiurewicz point. If three parameter rays accumulate at a common combinatorial class, then this combinatorial class is either a Misiurewicz point or a hyperbolic component, and the three rays land.

Proof. We will argue similarly as in Corollary 4.6. Denote the three external angles by $\vartheta_1 < \vartheta_2 < \vartheta_3$ and let Y be their common fiber. Choose preperiodic external angles $\alpha_1 \in (\vartheta_1, \vartheta_2)$ and $\alpha_2 \in (\vartheta_2, \vartheta_3)$. The corresponding parameter rays $R_{\mathcal{M}}(\alpha_i)$ land at two Misiurewicz points c_1 and c_2. Applying the Branch Theorem 3.1 to c_1 and c_2, we find either that one of these two points separates the other from the origin, or there is a Misiurewicz point or hyperbolic component which separates both from each other and from the origin. In all cases, it is impossible to connect the three parameter rays at angles $\vartheta_{1,2,3}$ to a single fiber, unless these three rays land at the separating Misiurewicz point or hyperbolic component. In the Misiurewicz case, all three angles must be preperiodic by Corollary 7.2. In the hyperbolic case, the landing points of the three rays $R_{\mathcal{M}}(\vartheta_i)$ have trivial fibers by Corollary 6.3, so all three rays must land at a common point. This is impossible by Corollary 6.5.

If, however, the three rays are only required to land at a common combinatorial class, then this combinatorial class may either be a Misiurewicz point or a hyperbolic component, and both cases obviously occur. □

We have defined fibers of a Multibrot set $\mathcal{M}_d$ using parameter rays at periodic and preperiodic angles. However, it turns out that only periodic angles are necessary.

Proposition 8.6 (Fibers Using Periodic Parameter Rays).
Fibers for a Multibrot set $\mathcal{M}_d$ remain unchanged when they are defined using only parameter rays at periodic angles, rather than at all rational angles.

Proof. Preperiodic parameter rays never land on the boundary of an interior component of $\mathcal{M}_d$, so separation lines using preperiodic rays are always ray pairs. We only have to show that if any two parameters can be separated by a preperiodic ray pair, then there is a periodic ray pair separating them as well. In Theorem 7.3, we have shown that the fiber of any Misiurewicz point is trivial even when only periodic parameter ray pairs are used.

Now let c_1 and c_2 be two parameters of $\mathcal{M}_d$ in different fibers. If they are separated by a parameter ray pair at preperiodic angles, let c_0 be the Misiurewicz point at which this ray pair lands. Then there is a periodic ray pair separating c_0 and c_1. Since the rays landing at c_0 separate c_1 and c_2, this periodic ray pair must also separate c_1 and c_2 and we are done. □

It follows that combinatorial classes are exactly the pieces that we can split Multibrot sets into when using only periodic ray pairs as separation lines. This is exactly the partition used to define internal addresses in [**LS1**] (compare also [**BrS**]), so combinatorial classes are the objects where internal addresses live naturally.

THEOREM 8.7 (The Pinched Disk Model of Multibrot Sets).
The quotient of any Multibrot set in which every fiber is collapsed to a point is a compact connected locally connected Hausdorff space.

PROOF. Every fiber of $\mathcal{M}_d$ is closed. In fact, the entire equivalence relation is closed: suppose that (z_n) and (z'_n) are two converging sequences in $\mathcal{M}_d$ such that z_n and z'_n are in a common fiber for every n. The limit points must then also be in a common fiber: if they are not, then they can be separated by either a ray pair at periodic angles or by a separation line through a hyperbolic component, and the separation runs in both cases only through points with trivial fibers. In order to converge to limit points on different sides of this separation line, all but finitely many points of the two sequences must be on the respective sides of the separation line, and z_n and z'_n cannot be in the same fiber for large n.

It follows that the quotient space is a Hausdorff space with respect to the quotient topology. It is obviously compact and connected. Local connectivity follows from a similar proof as in Proposition 4.5 where we showed that trivial fibers imply local connectivity. □

This quotient space is called the "pinched disk model" of the Multibrot set; compare Douady [**Do**]. It comes with an obvious continuous projection map π from the actual Multibrot set, and inverse images of points under π are exactly fibers of $\mathcal{M}_d$. Closely related is Thurston's lamination model [**T**].

In closing, we should mention that triviality of many more fibers is known, mostly for the Mandelbrot set $\mathcal{M}_2$: Yoccoz [**H**, Part III] has shown that the fiber of any $c \in \mathcal{M}_2$ is trivial if c is not infinitely renormalizable (that is, c is not contained in an infinite sequence of nested copies of Mandelbrot sets within itself); for certain parameters, the proof generalizes to degrees $d > 2$, while for others it does not. Lyubich [**L**] has established triviality of many further fibers, and the surgery methods of Riedl [**R**] combined with [**S4**] allow to transfer triviality of fibers from given parameters to many others, leading to new results mainly for $d > 2$.

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